Random Matrices, Large Deviations and Reflected Brownian Motion

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Abstract

In this thesis we present results in large deviations theory, free probability and the theory of reflected Brownian motion.

We study the large deviations behaviour of the block structure of a non-crossing partition chosen uniformly at random. This allows us to apply the free moment-cumulant formula of Speicher to express the spectral radius of a non-commutative random variable in terms of its free cumulants.

Next the distributions of three quadratic functionals of the free Brownian bridge are studied: the square norm, the signature and the Lévy area of the free Brownian bridge. We introduce two representation of the free Brownian bridge as series involving free semicircular variables, analogous to classical results due to Lévy and Kac. The latter representation extends to all semicircular processes. For each of the three quadratic functionals we give the R-transform, from which we extract information about the distribution, including free infinite divisibility and smoothness of the density. We also apply our result about the spectral radius to compute the maximum of the support for Lévy area and square norm. In both cases we obtain implicit equations.

The final chapter of the thesis is devoted to the study of a generalisation of reflected Brownian motion (RBM) in a polyhedral domain. This is motivated by recent developments in the theory of directed polymer and percolation models, in which existence of an invariant measure in product form plays a role. Informally, RBM is defined by running a standard Brownian motion in the polyhedral domain and giving it a singular drift whenever it hits one of the boundaries, kicking the process back into the interior. Our process is obtained by replacing this singular drift by a continuous one, involving a continuous potential. RBM has an invariant measure in product form if and only if a certain skew-symmetry condition holds. We show that this result extends to our generalisation. Applications include examples motivated by queueing theory, Brownian motion with rank-dependent drift and a process with close connections to the δ-Bose gas.

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Declaration

The author declares that the material contained in this thesis is entirely his own work, with the exception of the material in Chapter 5, which arose from joint work with Neil O'Connell. As is standard practice, the subject matter developed builds on existing theory, and clear citations and references are provided where necessary.

No part of this thesis has been submitted, for the purposes of a degree or otherwise, to any other university or educational institution.

The material in Chapter 3 has been accepted for publication to the *Electronic Journal of Probability*, under the title *Large deviations for non-crossing partitions* [91].

Chapter 4 is based on the paper *Functionals of the free Brownian bridge* [90] by the author, which has been submitted to the *Séminaires de probabilités.*

The results in Chapter 5 will appear in the paper *Product-form invariant measures for Brownian Motion with drift satisfying a skew-symmetry type condition* [87].

List of Notations and Abbreviations

$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_0$	the set of non-negative integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
$D\,(I, E)$	the space of càdlàg functions from an interval $I \subset \mathbb{R}$ to a topological space E.
$\mathcal{C}(E)$	the space of continuous real-valued functions on a topological space E
$\mathcal{C}_b(E)$	the subspace of $\mathcal{C}(E)$ consisting of all bounded functions
S^1	the unit circle in $\mathbb{R}^2$
$\mathfrak{M}_+(E)$	the set of positive finite measures on a topological space E with its Borel σ-algebra
$\mathfrak{M}_s(E)$	the subset of $\mu \in \mathfrak{M}_+(E)$ such that $\mu(E) = s$, for some $s \in (0, \infty)$
δ_x	the Dirac measure concentrated on x
$m_k(\mu)$	the k^{th} moment of a measure μ on $\mathbb{R}$.
$\lvert x \rvert$	the Euclidean norm of a vector $x \in \mathbb{R}^d$.
i.i.d.	independent and identically distributed
$\tau(V', V)$	the coarsest topology on the algebraic dual of a vector space V' with respect to which the maps $f \longmapsto f(x)$ are continous for all $x \in V$.

Unless otherwise stated we equip sets of measures with the weak topology.

Chapter 1

Introduction

The theory of free probability was introduced by VOICULESCU [123, 124, 125], originally as a tool in operator algebras, in order to study type II von Neumann algebras. Its two central ingredients are the concept of a non-commutative probability space on the one hand and a new type of independence, based on free rather than tensor products, on the other.

Probabilists' interest stems from the fact that random matrices can naturally be considered as non-commutative random variables and that many ensembles studied in random matrix theory are *asymptotically free*, that is, they become free as the size of the matrix tends to infinity. This allows one to compute joint asymptotic spectral distributions.

More recently free probability theory has also proved a useful tool in wireless communications [64, 77, 79, 118, 119], quantum information theory [5, 7] and the study of randomly disordered systems, in connection with the Anderson model [80, 110].

In all of these areas the *R-transform* is a powerful tool for computations. Given an R-transform we can, at least in theory, obtain the corresponding probability measure. However in order to do so one needs to find the functional inverse of the R-transform for which a closed-form expression may not exist. This raises the question what information can be inferred from the R-transform without inverting it. In Chapter 3 we present a formula for the right edge of the support of a probability measure in terms of its free cumulants.

As a key ingredient we establish a large deviations principle for the block structure of uniformly random non-crossing partitions. A law of large numbers, stating that the proportion of blocks of size k tends to 2^{-k} as the size n of the set to be partitioned goes to infinity, follows. This result in random combinatorics, in the spirit of [27, 35, 121], can be extended, via well-chosen bijections, to other combinatorial structures, including the descents of Dyck paths, the lengths of chains in ordered trees and the blocks of non-nesting partitions.

The large deviations result allows us to apply Varadhan's lemma to SPEICHER's free moment-cumulant formula in order to express the right edge of the support of a probability measure μ as a variational formula involving its free cumulants, provided these are non-negative.

The semicircle law, which arises as the asymptotic spectral distribution of Wigner matrices [129] is in many ways the free analogue of the Gaussian distribution. We have the concept of a semicircular process, of which the *free Brownian motion* is a prominent example. It can be considered as the limit of Brownian motion on the space of $N \times N$ Hermitian matrices as the size N tends to infinity. From the free Brownian motion we obtain the *free Brownian bridge* in the same way as in classical probability theory. In Chapter 4 we introduce two representations of the free Brownian bridge as series of freely independent semicircular random variables, one of which extends easily to all semicircular processes.

These representations allow us to prove various properties of three quadratic functionals of the free Brownian bridge: it's L^2-norm, its signature and its Lévy area. In each case we compute the R-transform and show free infinite divisibility. For L^2-norm and Lévy area we show that the underlying distributions have smooth densities with respect to Lebesgue measure and give implicit equations for the density. Using the spectral radius result from Chapter 3 we also compute the right edge of the support. For the signature a connection with certain combinatorial objects called *irreducible meanders* appears somewhat unexpectedly.

There has been much recent development on the study of positive-temperature percolation and polymer models [33, 84, 88, 103, 104]. These turn out to be closely related to random matrix theory on the one hand and the study of the *KPZ equation* [58], which was proposed to describe a class of surface growth models, on the other. An important role is played by an exactly solvable discrete directed percolation model which was introduced in [89] and further studied in [26, 76, 84, 104, 111]. In [84] the partition function of the polymer model is related to a diffusion process whose generator is given in terms of the Hamiltonian of the *quantum Toda lattice*. The proof uses a multi-dimensional generalisation of results by MATSUMOTO–YOR [68, 69, 71], concerning exponential functionals of Brownian motion. For connections between these models, Whittaker functions and representation theory we refer to the recent survey [86].

The polymer model introduced in [89] can also be viewed as a network of *generalised Brownian queues* in tandem. A crucial role is played by an analogue of the *output* or *Burke theorem*, which states that the output of the M/M/1 queue up to a fixed time t is independent of the queue length at time t. This leads to a product-form invariant distribution for the series of queues.

Queueing networks [34, 50, 51, 96] provide examples of *reflected Brownian motion (RBM)* in a polyhedral domain, introduced and studied by HARRISON and WILLIAMS [53, 130]. The analogue of the output theorem in this setting is the existence of an invariant measure in product form. The main result in [130] is that RBM in a polyhedral domain has an invariant measure in product form if and only if a certain *skew-symmetry condition* holds.

Motivated by this we introduce a multidimensional diffusion we call *generalised RBM (GRBM)*. Rather than giving the Brownian motion a singular drift whenever it hits one of the boundaries, we now impose a continuous drift. Its magnitude depends, via a potential U, on the position of the process relative to the polyhedral domain. A special case, the *exponentially RBM*, corresponds to the choice $U(x) = -e^{-x}$, which corresponds to the generalised Brownian queue.

We show that for the GRBM existence of an invariant measure in product form is still equivalent to the skew-symmetry condition of Harrison–Williams, independent of the function U.

By introducing a parameter β (which can be viewed as *inverse temperature*) and letting $\beta \longrightarrow \infty$, we recover the diffusion studied by Harrison–Williams. In this sense our process really is a generalisation of reflected Brownian motion.

Apart from examples motivated by queueing networks we also study the analogue of *Brownian motion with rank-dependent drift* [92] and draw connections to the δ-*Bose gas* recently studied and related to reflected Brownian motion by Prolhac–Spohn [95].

Chapter 2

Background Material

This chapter is devoted to those aspects of combinatorics, large deviations theory, free probability theory and reflected Brownian motion that we require later on.

2.1 Catalan Structures and Combinatorics

We present here some background on combinatorics, in particular Catalan structures. Our focus will lie on non-crossing partitions and Dyck paths.

2.1.1 Dyck Paths and Non-Crossing Partitions

Definition 2.1.1. A partition π of the set $\underline{n} = \{1, \ldots, n\}$ is said to be *crossing* if there exist distinct blocks V_1, V_2 of π and $x_j, y_j \in V_j$ such that $x_1 < x_2 < y_1 < y_2$. Otherwise π is said to be *non-crossing*. Equivalently, label the vertices of a regular n-gon $1, \ldots, n$ then π is non-crossing if and only if the convex hulls corresponding to the blocks are pairwise disjoint. We denote the set of all non-crossing partitions of $\underline{n}$ by $\mathrm{NC}(n)$.

Non-crossing partitions were first introduced by G. Kreweras [60] and has first attracted attention from combinatorialists. Later they have also been studied in connection with low-dimensional topology and geometric group theory, symmetric groups [72], algebraic combinatorics and mathematical biology [93, 105]. We will explore in some detail further connections with parking functions and free probability theory.

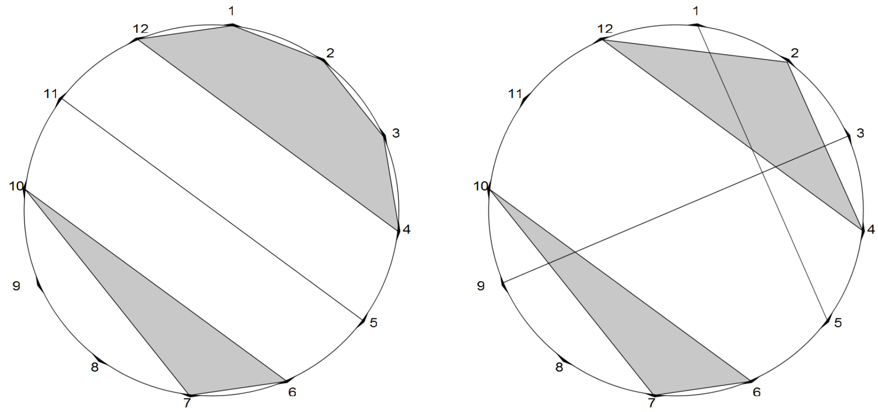

Figure 2.1: The partition $\{\{8\},\{9\},\{10,7,6\},\{11,5\},\{12,4,3,2,1\}\}$ is non-crossing, $\{\{5,1\},\{8\},\{9,3\},\{10,7,6\},\{12,4,2\}\}$ is crossing.

A *Dyck path* of *semilength* n is a lattice path in $\mathbb{Z}^2$ that never falls below the horizontal axis, starting at $(0,0)$ and ending at $(2n,0)$, consisting of steps $(1,1)$ (*upsteps*) and $(-1,1)$ (*downsteps*). Every such path consists of exactly n up- and downsteps each. The set of Dyck paths of semilength n is denoted by $\mathcal{P}(n)$. A maximal sequence of upsteps is called an *ascent*, while a maximal sequence of downsteps is referred to as a *descent*.

The cardinalities of $\mathcal{P}(n)$ and $\mathrm{NC}(n)$ are both given by the n^{th} *Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

There is a remarkably large collection of combinatorial objects with this property, usually referred to as *Catalan structures*. These include triangulations of convex

$(n+2)$-gons, binary trees with n vertices and pairs of standard Young tableaux of the same shape, consisting of n squares and at most 2 rows. A long list of examples was compiled by Stanley [114] (Exercise 6.1.9), where many results and references on Catalan structures can also be found.

There is a well-known bijection $\Phi\colon \mathcal{P}(n) \longrightarrow \mathrm{NC}(n)$ which maps the descents of $p \in \mathcal{P}_n$ to the blocks of $\Phi(p)$ [30, 131]. Given $p \in \mathcal{P}_n$ label the upsteps from left to right by $1, \dots, n$. Label each downstep by the same index as its corresponding upstep, that is the first upstep to the left on the same horizontal level. Then the descents induce an equivalence relation on $\underline{n}$: two labels are equivalent if and only if the corresponding downsteps are part of the same descent. The associated partition is then easily seen to be non-crossing.

Conversely, given $\pi = \{V_1, \dots, V_r\} \in \mathrm{NC}(n)$ write the elements of each block V_j in descending order, then sort the blocks in ascending order by their largest elements. This gives the descent structure of $\Phi^{-1}(\pi)$, which can be complemented by ascents in a unique way to form a Dyck path.

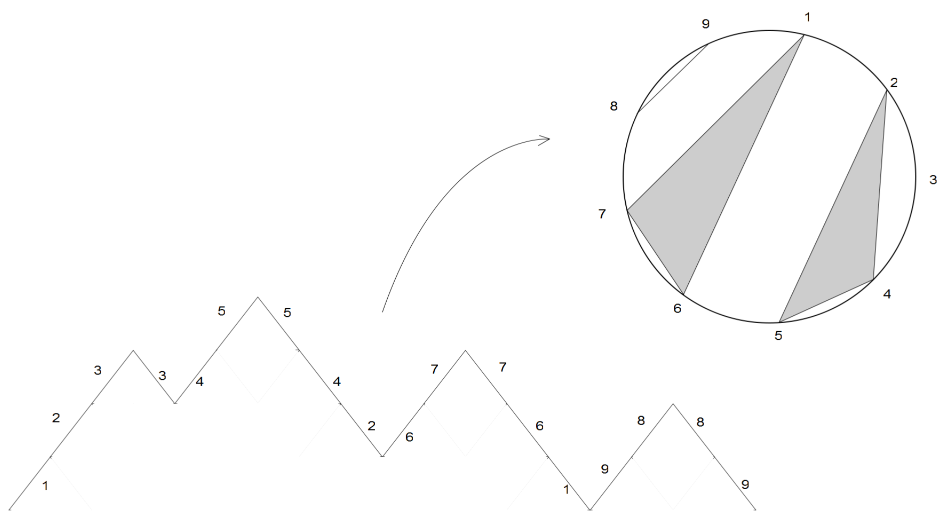

Figure 2.2: An example for the bijection Φ.

Given the vast number of Catalan structures it is not surprising that there exist other bijections which map the blocks of a non-crossing partition to an interesting substructure. We mention here the lengths of chains in ordered trees [94] and the blocks of non-nesting partitions [97].

As a small generalisation let us mention the k-divisible non-crossing partitions for some $k \in \mathbb{N}$. A non-crossing partition of the set $\underline{m}$ is said to be *k-divisible* if the size of each block is divisible by k. Of course such a partition can only exist if m is a multiple of k and we denote by $\mathrm{NC}^{(k)}(n)$ the set of all k-divisible non-crossing partitions of kn. The image of $\mathrm{NC}^{(k)}(n)$ under the bijection Φ described above can be identified [6] with the set of *k-Dyck paths*, i.e. the paths with upsteps $(1,1)$ and downsteps $(1,-k)$. The cardinality of $\mathrm{NC}^{(k)}(n)$ is given [44] by the *Fuss-Catalan numbers*

$$\mathrm{NC}^{(k)}(n) = \frac{1}{n}\binom{(k+1)n}{n-1}.$$

See Armstrong [3] for a survey of a more general object on Coxeter groups.

2.1.2 Lattices, Parking Functions and Permutations

The set of non-crossing partitions can be given a partial order $\preceq$ (called the *reverse refinement order*) defined by setting $\pi \preceq \sigma$ if and only if every block of π is completely contained in one of the blocks of σ. The maximal element of $\mathrm{NC}(n)$ with respect to this order is the partition 1_n with a single equivalence class. The partition consisting of n singleton blocks, denoted 0_n, is the minimal element. By Proposition 9.17 in Nica–Speicher [83], the partial order $\preceq$ induces a *lattice structure* on $\mathrm{NC}(n)$, that is for any $\pi, \sigma \in \mathrm{NC}(n)$ there exists

(i) a *join* $\pi \vee \sigma$, that is an element $\tau \in \mathrm{NC}(n)$ such that $\tau \succeq \sigma$ and $\tau \succeq \pi$ that has $\tau \preceq \tau'$ for any other τ' with that property

(ii) a *meet* $\pi \wedge \sigma$, i.e. $\tau \succeq \sigma, \mu$ that is the largest with this property.

The lattice of non-crossing partitions can also be embedded into the Cayley graph of the symmetric group S_n. This map, due to BIANE [17] allows us to relate our large deviations result to the cycle structure of the permutations lying on a geodesic from the identity element in S_n to the maximal cycle $(1 \ldots n)$.

A *geodesic* between two points a, b on any non-oriented graph $G = (V, E)$ is a path from a to b in G of minimal length. For vertices v_1, v_2 we denote by $[v_1, v_2]$ the set of vertices of G that lie on some geodesic from v_1 to v_2. This is an ordered set, indeed another lattice, with $w_1 \leq w_2$ if and only if w_1 lies on some geodesic from v_1 to w_2.

If G is the Cayley graph of S_n with the collection of all transpositions as generator set then [17] gives an order-preserving bijection Ψ from $[e, (1 \ldots n)]$ to $\mathrm{NC}(n)$. The map Ψ is given by associating to a permutation the partition given by its cycle structure. In particular Ψ maps bijectively the cycle structure of $[e, (1 \ldots n)]$ to the block structure of $\mathrm{NC}(n)$, so we get a large deviations principle, of speed n and with rate function J given above, for the cycle structure of uniformly random elements of $[e, (1 \ldots n)]$.

In [18] P. BIANE uses this bijection to re-derive a bijection between maximal chains in the lattice $\mathrm{NC}(n+1)$ and parking functions on n, originally established by STANLEY [113]. A *parking function* is a sequence of natural numbers $(a_1, \ldots, a_n)$ such that its increasing rearrangement $(a_{(1)}, \ldots a_{(n)})$ has $a_{(j)} \leq j$ for all j.

2.1.3 Block Structure for Non-Crossing Partitions

We survey here previously known enumerative and asymptotic results about the block structure of non-crossing partitions. For a given $\pi \in \mathrm{NC}(n)$ we denote by $B_k(\pi)$ the number of blocks of size k in π, and by $B(\pi)$ the total number of blocks of π.

The number of elements $\pi \in \mathrm{NC}(n)$ which have r_j blocks with j elements for each $j \in \underline{n}$ is given [60] by

$$\frac{n!}{\prod_{j=1}^{n} r_j! \left(n+1-\left(\sum_{j=1}^{n} r_j\right)\right)!}.$$

The number of non-crossing partitions of $\underline{n}$ with k blocks is given [40] by the *Narayana numbers*

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Therefore the expected number of descents in w is $\frac{n+1}{2}$. One could also deduce this latter fact using the *Kreweras complement* $K\colon \mathrm{NC}(n) \longrightarrow \mathrm{NC}(n)$, which is defined as follows: consider additional numbers $\overline{1}, \ldots, \overline{n}$ and interlace them with $1, \ldots, n$ so that $\overline{j}$ lies between j and $j+1$. Then the Kreweras complement $K(\pi)$ of $\pi \in \mathrm{NC}(N)$ is the biggest element (with respect to the inverse refinement order) of those $\sigma \in \mathrm{NC}(\overline{1}, \ldots, \overline{n})$ with the property that the partition $\pi \cup \sigma$ of the set $\{1, \overline{1}, \ldots, n, \overline{n})$, formed by taking all the blocks of π and all those of σ together, is non-crossing. Then [83] K is a lattice anti-isomorphism (i.e. a bijection such that $\sigma \preceq \pi$ implies $K(\sigma) \succeq K(\pi)$ such that for any $\pi \in \mathrm{NC}(n)$,

$$B(\pi) + B\left(K\left(\pi\right)\right) = n+1.$$

From this it follows directly that the expected number of blocks is $\frac{n+1}{2}$.

Let further $T(n,k)$ denote the number of $\pi \in \mathrm{NC}(n)$ with k singleton blocks and denote its generating function by $G(t,z)$, then [131, p.3153]

$$G(t,z) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} T(n,k) t^k z^n = \frac{1+(1-t)z-\sqrt{1-2(1+t)z+(2t+t^2-3)z^2}}{2z\left(1+(1-t)z\right)}.$$

Denote the total number of singletons in all $\pi \in \mathrm{NC}(n)$ by α_n, then

$$\sum_{n=1}^{\infty} \alpha_n z^n = \sum_{n=1}^{\infty}\sum_{k=0}^{n} kT(n,k)z^n = \frac{\partial}{\partial t}G(t,z)\bigg|_{t=1} = \frac{z}{\sqrt{1-4z}} = \sum_{n=0}^{\infty} \begin{pmatrix} 2n \\ n \end{pmatrix} z^{n+1}.$$

So the expected number of singletons in a non-crossing partition chosen uniformly at random is given by

$$\frac{1}{C_n}\begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} = \frac{n^2+n}{4n-4} \simeq \frac{n}{4}.$$

Asymptotically we therefore have about $\frac{n}{2}$ descents, roughly half of which are singletons. This might suggest that about half of the remaining descents is of length 2 and so on, and indeed we will see in Chapter 3 below that this is the case.

2.2 Free Probability Theory

We dicuss here some concepts and results from free probability theory. More on the subject can be found, for example, in the books [55, 127, 128] and the survey aimed at probabilists by Biane [19]. For the concepts in classical probability that we mention we refer to the books [98, 100, 101].

2.2.1 Non-Commutative Probability Spaces

Before defining the basic object of study in free probability we need to recall some notions from functional analysis. More details can be found in the book by Murphy [78].

Definition 2.2.1. A complex algebra $\mathcal{A}$ with unit $\mathbf{1}_{\mathcal{A}}$ and an involution ${}^*\colon \mathcal{A} \longrightarrow \mathcal{A}$ such that $\mathbf{1}_{\mathcal{A}}^* = \mathbf{1}_{\mathcal{A}}$ is said to be a $*$-*algebra*. A *-algebra equipped with a norm $\|\cdot\|$ is called a C^*-*algebra* if $\mathcal{A}$ is complete with respect to $\|\cdot\|$, $\|\mathbf{1}_{\mathcal{A}}\| = 1$ and for all $a, b \in \mathcal{A}$ we have

$$\|a^*\| = \|a\|, \quad \|a^*a\| = \|a\|^2, \quad \|ab\| \leq \|a\|$$

Elements $a \in \mathcal{A}$ are said to be *normal* if $a^*a = aa^*$, *self-adjoint* if $a^* = a$ and *positive* if there exists self-adjoint $b \in \mathcal{A}$ such that $a = b^2$.

Definition 2.2.2. A *non-commutative probability space* is a *-algebra $\mathcal{A}$ together with a *state* ϕ, that is a linear functional $\phi\colon \mathcal{A} \longrightarrow \mathbb{C}$ such that $\phi(\mathbf{1}_{\mathcal{A}}) = 1$ and $\phi(a) \geq 0$ whenever a is positive.

Whenever topological concepts like convergence are involved we will additionally assume that $(\mathcal{A}, \phi)$ is a C^*-*probability space*, that is $\mathcal{A}$ is a C*-algebra and ϕ continous. We will always implicitly assume ϕ to be *tracial* (meaning $\phi(ab) = \phi(ba)$ for all $a, b \in \mathcal{A}$) and *faithful* (i.e. $\phi(a^*a) = 0$ implies $a = 0$).

We think of elements $a \in \mathcal{A}$ as non-commutative random variables and consider $\phi(a)$ to be the expectation of $a \in \mathcal{A}$. This is motivated by the fact that for a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the space $\mathcal{A}$ of bounded complex-valued random variables equipped with the state $\mathcal{A} \ni X \longmapsto \mathbb{E}(X) \in \mathbb{C}$ is a non-commutative probability space in the sense of our definition.

If $a \in \mathcal{A}$ is self-adjoint there exists a compactly supported measure μ_a on $\mathbb{R}$, called the *distribution* of a, such that

$$\phi(a^n) = \int t^n \, \mu_a(\mathrm{d}t) \qquad \forall \, n \in \mathbb{N}.$$

The existence of a distribution is analogous to classical probability theory. We note, however, that in our setting the distribution is always compactly supported. In this sense we are therefore only considering bounded random variables. There is a theory of unbounded non-commutative random variables, using affiliated elements of operator algebras, but we do not require this here and instead refer to the papers by Maassen [67] and Bercovici–Voiculescu [13].

Example 2.2.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space, fix $N \in \mathbb{N}$ and let $\mathcal{A}_N$ be the algebra of $N \times N$ *random matrices with all moments*, that is $N \times N$ matrices whose entries are elements of

$$L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}) = \bigcap_{p \geq 1} L^p(\Omega, \mathcal{F}, \mathbb{P}).$$

Equipped with the state ϕ_N given by $\phi_N(a) = \mathbb{E}\mathrm{tr}_N(a)$ (where tr_N denotes the normalised trace which maps the identity matrix to 1) this forms a non-commutative probability space. If $a \in \mathcal{A}$ is normal then the spectral theorem guarantees the existence of N (not necessarily distinct) eigenvalues $\lambda_1, \ldots, \lambda_N$ of a. The distribution

μ_a of a is then characterised by its action on continuous test functions $f \in \mathcal{C}(\mathbb{R})$:

$$\int f \mathrm{d}\mu_a = \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}\left[f(\lambda_j)\right].$$

Note that if $a \in \mathcal{A}_N$ is self-adjoint the eigenvalues $\lambda_1, \ldots, \lambda_N$ of a are real and hence the support of μ_a is a subset of the real line.

Example 2.2.4. Let $\mathcal{H}$ be a complex separable Hilbert space and fix $h \in \mathcal{H}$ with $\|h\| = 1$. Let $\mathcal{A} = B(\mathcal{H})$, the C*-algebra of bounded operators on $\mathcal{H}$. Equipped with the faithful normal trace ϕ given by

$$\phi(a) = \langle a(h), h \rangle$$

the pair $(\mathcal{A}, \phi)$ is a non-commutative probability space. In fact, by the GNS construction [78], every non-commutative probability space can be realised in this way. See also Section 2.2.6.

2.2.2 Freeness and Combinatorics

Having introduced non-commutative probability spaces we now turn to the second ingredient in free probability: free independence. We discuss this following the combinatorial approach by Speicher [107, 108, 109], relying on the presentation by Biane [19]. See also Nica–Speicher [81, 83]. An analytic approach to free probability will be presented in Section 2.2.3 below.

Definition 2.2.5. C*-subalgebras $\mathcal{B}_1, \ldots, \mathcal{B}_N$ of $\mathcal{A}$ are said to be *freely independent*, or *free*, if for every set of indices $\{r_j\}_{j=1}^m \subseteq \{1, \ldots, N\}$ and collection $\{a_j \in \mathcal{B}_{r_j} : 1 \leq j \leq m\}$ such that $r_j \neq r_{j+1}$ and $\phi(a_j) = 0 \;\forall\, j$ we already have

$$\phi(a_1, \ldots, a_m) = 0.$$

Random variables $a_1, \ldots, a_N$ are said to be free if the unital C*-algebras generated by the a_j are free.

Suppose $a_1, a_2 \in \mathcal{A}$ are free. Then we can calculate directly the joint moments of the a_j. For example:

$$\phi(a_1 a_2) = \phi(a_1)\phi(a_2) \tag{2.2.6}$$

$$\phi(a_1 a_2 a_1 a_2) = \phi(a_1)^2 \phi(a_2^2) + \phi(a_1^2)\phi(a_2)^2 - (\phi(a_1)\phi(a_2))^2 . \tag{2.2.7}$$

In fact, all joint moments of a_1, a_2 are determined by the restriction of ϕ to the subalgebras $\mathcal{A}_1 = \mathbf{A}(1, a_1)$, $\mathcal{A}_2 = \mathbf{A}(1, a_2)$. This compares well with the fact that we can calculate the joint distribution of an independent family of independent classical random variables given their marginal distributions. However, trying to find explicit formulae for higher moments by hand quickly leads to complicated formulae. This was elegantly solved by combinatorial means by R. Speicher.

Definition 2.2.8. The *free cumulants* of $\mathcal{A}$ are defined to be the maps $k_n \colon \mathcal{A}^n \longrightarrow \mathbb{C}$ $(n \in \mathbb{N})$ defined indirectly by the following system of equations:

$$\phi(a_1, \ldots, a_n) = \sum_{\pi \in \mathrm{NC}(n)} k_\pi[a_1, \ldots, a_n] \tag{2.2.9}$$

where k_π denotes the product of cumulants according to the block structure of π. That is, if $V_1, \ldots, V_r$ are the components of $\pi \in \mathrm{NC}(n)$ then

$$k_\pi[a_1, \ldots, a_n] = k_{V_1}[a_1, \ldots, a_n] \ldots k_{V_n}[a_1, \ldots, a_n]$$

where, for $V = (v_1, \ldots, v_r)$ we just have $k_V[a_1, \ldots, a_n] = k_{|V|}[a_{v_1}, \ldots, a_{v_r}]$.

Note that (2.2.9) has the form $\phi(a_1, \ldots, a_n) = k_n[a_1, \ldots, a_n] +$ lower order terms, so that we can find the k_n inductively. Alternatively, (2.2.9) defines the k_n by Möbius inversion. This fits in a larger frameworks of multiplicative functions on lattices of partitions, which is explained in [83].

Because of the definition and the fact that ϕ is linear it follows that for each n the function k_n is n-multilinear in its arguments.

We will write $k_n(a)$ for $k_n(a, \ldots, a)$. The *R-transform* of a random variable $a \in \mathcal{A}$ is defined, as formal power series, by

$$R_a(z) = \sum_{n=0}^{\infty} k_{n+1}(a) z^n. \tag{2.2.10}$$

Theorem 2.2.11. *For $a_1, \ldots, a_n \in \mathcal{A}$ the following two conditions are equivalent:*

(i) $a_1, \ldots, a_n$ are free,

(ii) mixed cumulants vanish: for all $n \geq 2$ we have, whenever $i(1), \ldots, i(m) \in \underline{n}$ such that there exist $p, q \in \underline{n}$ with $i(p) \neq i(q)$,

$$k_m\left(a_{i(1)}, \ldots, a_{i(m)}\right) = 0.$$

Corollary 2.2.12. *If $a, b \in \mathcal{A}$ are free then for all $n \in \mathbb{N}$ we have*

$$k_n(a+b) = k_n(a) + k_n(b).$$

So, if a_1 and a_2 are free, then the distribution of $a_1 + a_2$ is uniquely determined by those of a_1 and a_2.

For each pair of probability measures μ_1, μ_2 with compact support there exist a non-commutative probability space $(\mathcal{A}, \phi)$ and free $a_1, a_2 \in \mathcal{A}$ such that the distribution of a_j is μ_j. So we get a binary operation on the space of compactly supported probability measures, called *free convolution* and denoted $\boxplus$.

As a special case of Definition 2.2.8 we get the following *moment-cumulant formula* relating the free cumulants and moments of a non-commutative random variable.

Proposition 2.2.13. *For $a \in \mathcal{A}$ we have the following formula:*

$$\phi(a^n) = \sum_{\pi \in NC(n)} k_\pi. \tag{2.2.14}$$

On the level of formal power series [112] we have the following relation between the *Cauchy transform* G_{μ_a} of μ_a,

$$G_{\mu_a}(z) = \int \frac{\mu(\mathrm{d}t)}{z-t} = \sum_{n=0}^{\infty} \phi(a^n) z^{-n-1}$$

and the R-transform of a:

Corollary 2.2.15. *Let $a \in \mathcal{A}$ and G, R denote its Cauchy and R-transforms respectively. Considering these as formal power series we have*

$$G\left(R(z) + \frac{1}{z}\right) = z.$$

2.2.3 Analytic Aspects and Transforms

In contrast to the combinatorial approach with generating functions described above we now describe certain analytic considerations, using holomorphic functions. Throughout this section fix an element a of a non-commutative probability space $(\mathcal{A}, \phi)$ and denote its distribution by μ_a.

Since μ_a is compactly supported, the Cauchy transform G_{μ_a} defines an analytic map from $\mathbb{C}^+$ into $\mathbb{C}^-$, which extends analytically to a neighbourhood U_a of ∞. Rather than as a formal power series we can now consider the right-hand side as a holomorphic power series, valid on U_a. We will also write G_a for G_{μ_a}.

In this analytic framework we can reformulate Theorem 2.2.15 as follows. Because of compact support of μ_a the definition of the R-transform (2.2.10) defines an analytic function on a neighbourhood of zero [55, Theorem 3.2.1]. Moreover the Cauchy transform G_a of a is locally invertible on a neighbourhood of infinity and the inverse K_a satisfies

$$K_a(z) = R_a(z) + \frac{1}{z}.$$

Remark 2.2.16. Using the continuity of ϕ and multilinearity of the cumulants we now obtain the following properties of the R-transform:

1. If a_n converges to a in the operator topology of $\mathcal{A}$ then there exists a neighbourhood U of zero where R_n, R are defined for all $n \in \mathbb{N}$ and $R_{a_n}(z) \longrightarrow R_a(z)$ as $n \to \infty$ for every $z \in U$.

2. If $a, b \in \mathcal{A}$ are free then $R_{a+b}(z) = R_a(z) + R_b(z)$

3. For $\lambda \in \mathbb{C}$ we have $R_{\lambda a}(z) = \lambda R_a(\lambda z)$.

So the R-transform plays the role of the logarithm of the Fourier transform in classical probability theory: it is linear, additive on (freely) independent random variables, and determines the underlying distribution. Moreover its Taylor coefficients are also related to the moments by summing over a lattice of partitions: in the classical case over all, in the free case over the non-crossing partitions.

However, in contrast to the Fourier transform, it is not straightforward to read off properties of μ from its R-transform. The most direct way is to find the Cauchy transform (by inverting $R_\mu(z) + \frac{1}{z}$) and the applying Stieltjes inversion [55, p. 94]:

Theorem 2.2.17. *Let μ be a compactly supported probability measure them*

$$\mu = \lim_{y \downarrow 0} \left[-\frac{1}{\pi} \mathfrak{Im} G_\mu(x + iy) \right]$$

where the limit is in the weak topology on $\mathfrak{M}_1(\mathbb{R})$. Moreover $t_0 \in \mathbb{R}$ is an isolated point of the support of μ if and only if G_μ can be extended meromorphically to a neighbourhood of t_0 such that this extension has a simple pole at t_0. When μ has a continuous density f with respect to Lebesgue measure then

$$f(x) = -\frac{1}{\pi} \lim_{y \downarrow 0} G_\mu(x + iy).$$

2.2.4 Semicircular Processes

Definition 2.2.18. A collection $\mathcal{S} = (s_j)_{j \in I}$ of non-commutative variables on $\mathcal{A}$ is said to be a *semicircular family* with *covariance* $(c(i,j))_{i,j \in I}$ if the cumulants are given by

$$k_\pi[s_{j_1}, \ldots s_{j_n}] = \begin{cases} \prod_{p \sim_\pi q} c(j_p, j_q) & \text{if } \pi \text{ is a pair partition} \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{S}$ consists of a singleton s_1 and $r = 2\sqrt{c(1,1)}$ then the distribution of s_1 is the *centred semicircle law of radius* r, that is the measure σ_r on $\mathbb{R}$ given by

$$\sigma_r(\mathrm{d}t) = \frac{2}{\pi r^2} \sqrt{r^2 - t^2}\; \mathbf{1}_{[-r,r]}(t)\, \mathrm{d}t.$$

In particular σ_2 is also called the *standard semicircle law* and non-commutative random variables with law σ_r (σ_2) are referred to as *(standard) semicirculars.*

The semicircle law plays a similar role to the Gaussian distribution on classical probability theory. In particular there exists a central limit theorem (see below), and the only non-vanishing free cumulant is the second: the R-transform of a non-commutative random variable whose distribution is the centred semicircle law of radius r is given by

$$R(z) = \frac{r^2}{4}\, z.$$

Recall that the centred Gaussian distribution is characterised by the fact that only the second classical cumulant is nonzero.

Moreover a collection of random variables with a joint semicircular law is determined by its covariance. To be more precise we recall the following

Proposition 2.2.19 (Nica–Speicher [83], Proposition 8.19)**.** *Let $(s_i)_{i\in I}$ be a semicircular family of covariance $(c(i,j))_{i,j\in I}$ and suppose I is partitioned by $I_1, \ldots, I_d$. Then the following are equivalent:*

1. *The collections $\{s_j : j \in I_1\}, \ldots, \{s_j : j \in I_d\}$ are free*
2. *We have $c(r,j) = 0$ whenever $r \in I_p$ and $j \in I_q$ with $p \neq q$.*

In particular $\{s_j : j \in I\}$ is a free family if and only if $C = (c(r,j))_{r,j\in I}$ is diagonal.

Definition 2.2.20. A process $(X(t))_{t\geq 0}$ on $\mathcal{A}$ is said to be a *semicircular process* if for every $t_1, \ldots, t_n \in [0,\infty)$, the set $(X(t_1), \ldots, X(t_n))$ is a semircular family.

By the considerations above the finite-dimensional distributions of a semicircular process are determined by the *covariance structure* of the process, i.e. by the function $C\colon [0,\infty)^2 \longrightarrow \mathbb{C}$ defined by

$$C(s,t) = \phi(X(s)X(t)).$$

.

2.2.5 Law of Large Numbers and Central Limit Theorem

In order to discuss asymptotic freeness we will need to make precise the meaning of joint convergence for several sequences of free random variables. Unlike when only a single variable is involved, we can no longer use measure theory (because of non-commutativity) and are forced to work with moments only. Recall that for a single non-commutative random variable convergence in distribution is equivalent to convergence in moments.

Let $\mathbb{C}\langle X_1, \ldots, X_N\rangle$ denote the algebra of non-commutative polynomials in the variables $X_1, \ldots, X_N$.

Definition 2.2.21. Let $a_1, \ldots a_N \in \mathcal{A}$ for some non-commutative probability space $\mathcal{A}$. The linear functional $\mu_{a_1,\ldots,a_N} \colon \mathbb{C}\langle X_1, \ldots, X_N\rangle \longrightarrow \mathbb{C}$ defined by

$$\mu_{a_1,\ldots,a_N}\left(P\big(x_1, \ldots, x_N\big)\right) = \phi\left(P\left(a_1, ldots, a_N\right)\right)$$

is called the *joint distribution* of $a_1, \ldots a_N$.

Definition 2.2.22. Let I be a finite set. For each $n \in \mathbb{N}$ let $\left\{a_j^{(n)} \colon j \in I\right\}$ be a family of random variables in a noncommutative probability space $(\mathcal{A}_n, \phi_n)$ with joint distribution μ_n. The family of random variables is said to *converge in (joint) distribution* to $a_1, \ldots, a_N$ for some non-commutative probability space $(\mathcal{A}, \phi)$ if the joint distribution of $a_1, \ldots, a_N$ is μ and

$$\lim_{n\to\infty} \mu_n(P) = \mu(P) \qquad \forall\, P \in \mathbb{C}\langle\{X_j \colon j \in I\}\rangle.$$

A law of large numbers for free random variables was established by Bercovici–Pata [11]. We state it here in the context of self-adjoint random variables with real, compact support and refer to [11] the statement in full generality.

Theorem 2.2.23. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of free identically distributed random variables whose common distribution μ is compactly supported in $\mathbb{R}$. Then the non-commutative law of $\frac{1}{n}\sum_{k=1}^{n} X_k$ converges weakly to a point mass at $m_1(\mu)$.*

Furthering the analogy of semicircular distribution in free probability to the Gaussian law in the classical theory we have the following free central limit theorem.

It can be verified directly using free cumulants. For details we refer to [128] or Chapter 8 of [83].

Theorem 2.2.24 (Free Central Limit Theorem). *Let $(a_j)_{j\in\mathbb{N}}$ be a free family of random variables such that*

(i) $\phi(a_j) = 0$ *for all* $j \in \mathbb{N}$*;*

(ii) $\sup_{j\in\mathbb{N}} \left|\phi(a_j^k)\right| < \infty$ *for all* $k \in \mathbb{N}$*;*

(iii) $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \phi\left(a_j^2\right) = r^2/4.$

Then the sequence $(s_n)_{n\in\mathbb{N}}$ of random variables defined by

$$s_n = \frac{1}{n} \sum_{j=1}^{n} a_j$$

converges in non-commutative distribution to the centred semicircle law of radius r.

2.2.6 The Full Fock Space, Creation and Annihilation

In order to deal with convergence issues it will be useful to choose a specific non-commutative probability space. Let $\mathcal{H}_0$ be an infinite-dimensional separable complex Hilbert space and define the *full Fock space* to be

$$\mathrm{T}\left((\mathcal{H}_0)\right) = \bigoplus_{n=0}^{\infty} \mathcal{H}_0^{\otimes n} \tag{2.2.25}$$

where by convention $\mathcal{H}_0^{\otimes 0} = \mathbb{C}\Omega$ for a distinguished unit vector Ω. Equip the C*-algebra $B(\mathrm{T}\left((\mathcal{H}_0)\right))$ of continuous linear functionals on $\mathrm{T}\left((\mathcal{H}_0)\right)$ with the tracial state ϕ given by

$$\phi(a) = \langle a(\Omega), \Omega\rangle. \tag{2.2.26}$$

Definition 2.2.27. For $h \in \mathcal{H}_0$ define the *creation* and *annihilation operators* to be $l(h)$ and $l^*(h)$ respectively where

$$l(h)(h_1 \otimes \ldots \otimes h_n) = h \otimes h_1 \otimes \ldots \otimes h_n \tag{2.2.28}$$

$$l^*(h)(h_1 \otimes \ldots \otimes h_n) = \langle h, h_1 \rangle h_2 \otimes \ldots \otimes h_n. \tag{2.2.29}$$

Let $s(h)$ be the self-adjoint element of $B(\mathrm{T}((\mathcal{H}_0)))$ defined by $s(h) = l(h) + l^*(h)$. The following result is Theorem 2.6.2 in [128].

Lemma 2.2.30. *Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in $\mathcal{H}_0$ and put $\xi_n = s(e_n)$.*

(i) If $\mathcal{A}$ denotes the sub-C-algebra of $B(\mathrm{T}((\mathcal{H}_0)))$ generated by $(\xi_n)_{n\in\mathbb{N}}$ then ϕ is a faithful tracial state on $\mathcal{A}$.*

(ii) The set $\{s(e_n) \colon n \in \mathbb{N}\}$ forms a semicircular family in $\mathcal{A}$ with covariance kernel $C(m,n) = \delta_{mn}$.

Since all of the results in this thesis are only concerned with the distributions of non-commutative probability spaces we can and will assume throughout that $\mathcal{A}$ is a C*-subalgebra of $B(\mathrm{T}((\mathcal{H}_0)))$, the space of bounded linear operators on the full Fock space, and that ϕ is as in (2.2.26). In particular, all semicircular random variables that appear later on will be defined in terms of the creation and annihilation operators.

2.2.7 Free Probability and Random Matrix Theory

In this section we draw the connection between free probability and random matrix theory. For background on random matrix theory we refer to the books by Mehta [74], Anderson–Guionnet–Zeitouni [2] and Blower [24], and the St Flour lecture notes by Guionnet [49].

Recall from Example 2.2.3 that $N \times N$ random matrices can be considered as non-commutative random variables.

Consider two independent $N \times N$ Hermitian random matrices A_N, B_N. We know the distributions of their eigenvalues

$$\mu_{A_N} = \frac{1}{N}\sum_{k=1}^{N} \delta_{\lambda_k(A_N)}$$
$$\mu_{B_N} = \frac{1}{N}\sum_{k=1}^{N} \delta_{\lambda_k(B_N)}$$

where $\lambda_k(C)$ denotes the k^{th} largest eigenvalue of the matrix C. However the eigenvalues of $A_N + B_N$ depend on more than μ_{A_N} and μ_{B_N}. In particular there is no convolution relationship. Under certain assumptions on the random variables defining A_N and B_N it happens that, as $N \to \infty$, random matrices A_N, B_N converge to free non-commutative random variables s_A, s_B. By freeness the distribution of $s_A + s_B$ then only depends on the distributions of s_A and s_B.

The observation that certain random matrix ensembles become asymptotically free is the beginning of the relationship between free probability theory and random matrices. It is due to Voiculescu [126]. For the sake of definiteness we state the precise result in the case of random matrices sampled from the *Gaussian Unitary Ensemble (GUE).*

Theorem 2.2.31. *Fix an integer N. For each $s \in \underline{N}$ let $(Y(s,n))_{n\in\mathbb{N}}$ be a sequence random matrices, each taken from the $n\times n$ GUE ensemble such that every matrix is independent from all the others. Let Δ_n denote the algebra of deterministic diagonal $n \times n$ matrices and $D(t,n) \in \Delta_n$ for each $(t,n) \in \underline{N} \times \mathbb{N}$ such that $(D(t,n))_{t\in\underline{N}}$ has a joint limit distribution as $n \to \infty$ and $\sup_n \|D(t,n)\| < \infty$ for each t. Then the sequence of $2N$-tuples of random variables*

$$(D(1,n), \dots, D(N,n), Y(1,n), \dots Y(N,n))_{n\in\mathbb{N}}$$

converges to a $2N-tuple$ of free non-commutative random variables, the second N

entries of which are distributed according to the semicircle law.

We express this by saying that the sequences $\{Y(s,n)\colon n\in\mathbb{N}\}$ are *asymptotically free*. See Chapter 4 of [128]. Similar results hold if we replace the GUE by the *Gaussian Orthogonal Ensemble (GOE)* or choose the unitary group with Haar measure. In fact, the crucial assumption is that the entries are centred, have second moment $\frac{1}{n}$ and m-th moment uniformly bounded by $O(n^{-m/2})$, see [128], Theorem 4.4.1.

In particular if we have a pair of sequences of GUE(n) matrices they will converge in moments to (s_1,s_2) where s_1,s_2 are free semicircular (non-commutative) random variables.

Example 2.2.32. Let $Y(n)$ be an element of the $n\times n$ GUE ensemble and $D(n)$ a self-adjoint diagonal $n\times n$ matrix such that $\mu_{D(n)}\longrightarrow\nu$ for $\nu\in\mathcal{P}_K(\mathbb{R})$. Choose $f\in\mathbb{C}(\mathbb{R};\mathbb{R})$ and let μ be the asymptotic spectral distribution of

$$X(n)=D(n)+f(Y(n)). \tag{2.2.33}$$

Theorem 2.2.31 implies that $\mu=\nu\boxplus f_*\mu_s$. (where μ_s denotes the standard semicircle distribution). Note that with the R-transform we have a convenient tool for computing free convolutions.

2.2.8 Free Brownian Motion and Bridge

In classical probability theory a *Brownian motion* is a process with stationary and independent increments, characterised by the property that fixed-time marginals are Gaussian. As we saw above, there exist analogues in free probability theory for all these concepts. This motivates the definition of a free Brownian motion.

Definition 2.2.34. A non-commutative process $X\colon[0,\infty)\longrightarrow\mathcal{A}$ is said to be a *free Brownian motion* if

(i) the distribution of $X(t)$ is a centred semicircular law with radius t;

(ii) $X(t) - X(s)$ is free from $\{X(r) \colon r \leq s\}$;

(iii) $X(t) - X(s)$ has the same distribution as $X(t-s)$.

It is clear that properties (i) – (iii) uniquely characterise the law (that is, the finite-dimensional distributions) of X. A concrete realisation of free Brownian motion can be constructed on the full Fock space [23, 32, 61, 106].

Let H_N be the space of $N \times N$ Hermitian matrices, equipped with the inner product

$$\langle A, B \rangle_N = \mathrm{tr}_N(AB).$$

Hermitian Brownian motion can be defined [20] as the centred Gaussian process [98] M_N on $\mathcal{H}_N$ with covariance

$$\mathbb{E}\left[\langle AM_N(t), BM_N(s)\rangle_N\right] = (s \wedge t)\,\mathrm{tr}_N(AB).$$

For any fixed t the random matrix $M_N(t)$ has the same distribution as a re-scaled GUE.

We can also consider M_N as a free stochastic process, taking values in the non-commutative probability space $(\mathcal{A}_N, \phi_N)$ of Example 2.2.3. Then it follows from asymptotic freeness for GUE random matrices that M_N converges in non-commutative distribution to a free Brownian motion. That is, there exists a free Brownian motion X such that for every $t_1, \ldots, t_k \geq 0$ the k-tuple of non-commutative random variables $(M_N(t_1), \ldots, M_N(t_k))$ converges in joint distribution to the k-tuple $(X(t_1), \ldots, X(t_k))$.

There is also a free multiplicative Brownian motion, which can similarly be considered as the non-commutative limit of Brownian motion on the unitary group. We will not consider this process here and instead refer to Biane [15] and Bercovici–Voiculescu [14].

Definition 2.2.35. A centred semicircular process $(\beta_T(t))_{t\in[0,T]}$ on $\mathcal{A}$ is said to be a *free Brownian bridge* on $[0,T]$ if its covariance structure is given by

$$\phi(\beta_T(s)\beta_T(t)) = s \wedge t - \frac{st}{T}.$$

Remark 2.2.36. In analogy with classical probability it can be easily checked that if β is a free Brownian bridge on $[0,1]$ and ξ_0 is a free standard semicircular free from $\{\beta(t)\colon t \in [0,1]\}$, then $X(t) = \xi_0 t + \beta(t)$ defines a free Brownian motion.

2.3 Large Deviations Theory

Large deviations theory can be described as the study of rare events, more precisely how their probability behaves asymptotically on an exponential scale. We present here some of the main ideas. Our presentation is based on that of Dembo–Zeitouni [37], in which proofs omitted here can be found. We also refer to the books by den Hollander [38] and Deuschel–Stroock [39]. See also Ellis [45] for applications in statistical mechanics.

Throughout this section let E be a *Polish space*, that is a complete separable metric space, equipped with its Borel sigma-algebra. It will be sufficient for us to consider only large deviations for measures on Polish spaces. Some of the results below hold true in greater generality, see in particular [37, 39].

We will often encounter the set $[0,\infty]$ which we equip with the topology of the one-point compactification of $[0,\infty)$. By definition we consider the infimum of the empty set to be $+\infty$.

2.3.1 Basic Notions

Definition 2.3.1. A function $I\colon E \longrightarrow [0,\infty]$ is said to be *lower semi-continuous* if for every $\alpha \in [0,\infty)$ the corresponding *level set* $\Psi_I(\alpha) = \{x \in E\colon I(x) \le \alpha\}$ is closed. Such a map is called a *rate function* if $I(x) \neq \infty$ for at least one $x \in E$. A rate function I is said to be *good* if the level sets $\Psi_I(\alpha)$ are all compact.

Definition 2.3.2. A sequence of measures $(\mu_N)_{N\in\mathbb{N}}$ taking values on a Polish space is said to satisfy a *large deviations principle* (*LDP*) of *speed* $a = (a_N)_{N\in\mathbb{N}}$ with rate function I if a is a strictly increasing sequence of positive real numbers diverging to infinity and

$$\liminf_{N\to\infty} \frac{1}{a_N} \log \mu_N(G) \geq -\inf_{x\in G} I(x) \tag{2.3.3}$$

$$\limsup_{N\to\infty} \frac{1}{a_N} \log \mu_N(F) \leq -\inf_{x\in F} I(x) \tag{2.3.4}$$

for every open set G and every closed set F. (2.3.3) and (2.3.4) are often referred to as the large deviations *lower bound* and *upper bound* respectively. If (2.3.4) holds for all compact subsets of E then $(\mu_N)_{N\in\mathbb{N}}$ is said to satisfy a *weak LDP*.

If $(X_N)_{N\in\mathbb{N}}$ is a sequence of random variables taking values in E and the sequence of distributions μ_N satisfies an LDP then we also say that the sequence $(X_N)_{N\in\mathbb{N}}$ satisfies the LDP.

Remark 2.3.5. Equations (2.3.3) and (2.3.4) are equivalent to the following two conditions, which are often easier to check:

(i) Lower bound: for every $x \in E$ and every measurable subset A of E whose interior contains x,

$$\liminf_{N\to\infty} \frac{1}{a_N} \log \mu_N(A) \geq -I(x). \tag{2.3.6}$$

Equation (2.3.6) emphasises the local nature of the LDP lower bound.

(ii) Upper Bound: For every $\alpha \in [0,\infty)$ and every measurable subset A of E such that $I(x) > \alpha$ for all $x \in \overline{A}$,

$$\limsup_{N\to\infty} \frac{1}{a_N} \log \mu_N \leq -\alpha. \tag{2.3.7}$$

Strenghtening a weak LDP to a full one requires that most of the mass of the probability measures is concentrated, on an exponential scale, on compact sets. To be more precise:

Definition 2.3.8. A family of probability measures μ_N is said to be *exponentially tight* if for every $\alpha \in [0,\infty)$ there exists a compact subset K_α of E such that

$$\limsup_{N\to\infty} \frac{1}{a_N} \log \mu_N \left(E \setminus K_\alpha\right) < -\alpha. \tag{2.3.9}$$

From Lemma 1.2.18 in [37] it follows that if $(\mu_N)_{N\in\mathbb{N}}$ is exponentially tight and satisfies a weak LDP with rate function I then I is good and a full LDP holds for $(\mu_N)_{N\in\mathbb{N}}$.

We will also need to know how the LDP behaves under certain inclusions, which is the content of the following simple result.

Proposition 2.3.10. *Let E be a Polish space and A a measurable subset such that $\mu_N(A) = 1$ for all $n \in \mathbb{N}$. Equip U with the subspace topology induced by E.*

(a) If A is closed in E and $(\mu_N)_{N\in\mathbb{N}}$ satisfies the LDP in A with a given rate function $I\colon A \longrightarrow [0,\infty]$ then $(\mu_N)_{N\in\mathbb{N}}$ also satisfies the LDP in E, of the same speed and with rate function $\widetilde{I}\colon E \longrightarrow [0,\infty]$ defined by

$$\widetilde{I}(x) = \begin{cases} I(x) & \text{if } x \in A \\ +\infty & \text{otherwise.} \end{cases}$$

(b) If $(\mu_N)_{N\in\mathbb{N}}$ satisfies the LDP in E with rate function I such that $I(x) = \infty$ whenever $x \notin A$ then the same LDP holds in A.

Note that in the set-up of Proposition 2.3.10 (b), A closed implies $I(x) = \infty$ for $x \notin A$.

One of the first applications of the LDP is that it allows one to compute the logarithmic asymptotics of exponential functionals, by what is now called Varadhan's Lemma. Let $(X_N)_{N\in\mathbb{N}}$ be a sequence of E-valued random variables and denote the law of X_N by μ_N.

Theorem 2.3.11 (Varadhan's Lemma). *Suppose that $(\mu_N)_{N\in\mathbb{N}}$ satisfies the LDP with speed $(a_N)_{N\in\mathbb{N}}$ and good rate function I and let $\phi\colon E \longrightarrow \mathbb{R}$ be a continuous function such that one of the follwing two conditions holds: either the* tail condition*,*

$$\lim_{M\to\infty} \limsup_{N\to\infty} \frac{1}{a_N} \log \mathbb{E}\left\{e^{a_N\phi(Z_N)}\, \mathbf{1}_{\{\phi(X_N)\geq M\}}\right\} = -\infty$$

or the moment condition*: there exists $\gamma > 1$ such that*

$$\limsup_{N\to\infty} \frac{1}{a_N} \log \mathbb{E}\left\{e^{\gamma a_N\,\phi(Z_N)}\right\} < \infty.$$

Then

$$\lim_{N\to\infty} \frac{1}{a_N} \log \mathbb{E}\left\{e^{a_N\phi(X_N)}\right\} = \sup\left\{\phi(y) - I(y)\colon y \in E\right\}. \tag{2.3.12}$$

2.3.2 Contraction principles

A natural question to ask is what kind of transformations preserve the LDP. The first main result we present is the *contraction principle*, asserting that if a sequence of measures satisfy an LDP then so do their push-forwards under a fixed continuous function. Recall that for a measure μ on X the *push-foward* under a function $f\colon X \longrightarrow Y$ is the measure on Y given by $\mu \circ f^{-1}(A) = \mu(f^{-1}(A))$.

A related result gives conditions under which we can draw the converse implication. Finally we describe an LDP for the sequence of *product measures* $(\mu_N \otimes \nu_N)_{N\in\mathbb{N}}$

provided both $(\mu_N)_{N\in\mathbb{N}}$ and $(\nu_N)_{N\in\mathbb{N}}$ satisfy an LDP. This allows us to build up large deviations principles on a 'larger' space, a theme to which we will return in Section 2.3.4.

Theorem 2.3.13 (Contraction Principle)**.** *Let $f\colon X \longrightarrow Y$ be a continuous function between Polish spaces. If a sequence of measures $(\mu_N)_{N\in\mathbb{N}}$ on X satisfies an LDP with good rate function I then the sequence $(\mu_N \circ f^{-1})_{N\in\mathbb{N}}$ satisfies the LDP of the same speed and with good rate function $\widehat{I}_f$ given by*

$$\widehat{I}_f(y) = \inf\left\{I(x)\colon x \in f^{-1}(\{y\})\right\}.$$

Note that the rate function I in Theorem 2.3.13 is assumed to be good. If I is not a good rate function then $\widehat{I}_f$ may fail to be a rate function.

If the function f is a continuous injection then we can deduce the LDP for $(\mu_N)_{N\in\mathbb{N}}$ from that of $(\mu_N \circ f^{-1})_{N\in\mathbb{N}}$, provided we have exponential tightness. More precisely:

Theorem 2.3.14 (Inverse contraction principle)**.** *Let X, Y be Polish spaces and $f\colon X \longrightarrow Y$ continuous and injective. Let further $(\mu_N)_{N\in\mathbb{N}}$ be an exponentially tight sequence of probability measures on X. If $(\mu_N \circ f^{-1})_{N\in\mathbb{N}}$ satisfies the LDP on Y with rate function I then $(\mu_N)_{N\in\mathbb{N}}$ satisfies the LDP of the same speed and with rate function $I \circ f$.*

We note that here goodness of the rate function is not part of the assumptions, while goodness in the *conclusion* of the rate function $I \circ g$ follows from exponential tightness.

We conclude this section with the following result on product measures. It can be found as Corollary 2.9 in Lynch–Sethuraman [65]. See Exercise 4.2.7 in [37] for a more general version.

Theorem 2.3.15. *Let $(\mu_N)_{N\in\mathbb{N}}$ and $(\nu_N)_{N\in\mathbb{N}}$ be exponentially tight sequences of probability measures on Polish spaces X, Y respectively, satisfying large deviations principles of the same speed $(a_N)_{N\in\mathbb{N}}$ and respective good rate functions I_1, I_2. Then the sequence of product measures $(\mu_N \otimes \nu_N)_{N\in\mathbb{N}}$ satisfies the LDP on $X \times Y$ of speed $(a_N)_{N\in\mathbb{N}}$ and with the good rate function $I\colon X \times Y \longrightarrow [0,\infty]$ given by*

$$I(x, y) = I_1(x) + I_2(y).$$

2.3.3 Cramér's Theorem and Sanov's Theorem

For a sequence $(X_N)_{N\in\mathbb{N}}$ of i.i.d. E-valued random variables we summarise here large deviations results for the associated *empirical means*, defined by

$$\overline{X}_N = \frac{1}{N}\sum_{k=1}^{N} X_k \in E \tag{2.3.16}$$

and *emprical measures*

$$L_N^{(X)} = \frac{1}{N}\sum_{k=1}^{N} \delta_{X_k} \in \mathfrak{M}_1(E). \tag{2.3.17}$$

We will also discuss extensions to path-wise results.

Let μ be a probability measure on a locally convex topological Hausdorff space. The *logarithmic moment generating function* is defined by the function $\Lambda_\mu\colon E^* \longrightarrow \mathbb{R}$ defined by

$$\Lambda_\mu(\lambda) = \log \int_E e^{\lambda}(x)\, \mu(\mathrm{d}x).$$

While $\Lambda_\mu(\lambda) > -\infty$ for all $\lambda \in \mathbb{E}^*$ it can happen that $\Lambda(\lambda) = +\infty$, and we denote the subset of those $\lambda \in E^*$ with $\Lambda(\lambda) \in \mathbb{R}$ (the *effective domain* of Λ) by D_{Λ_μ}. We also denote by Λ_μ^* the *Fenchel–Legendre* transform of Λ, that is

$$\Lambda_\mu^*(x) = \sup\left\{\lambda(x) - \Lambda_\mu(\lambda) \colon \lambda \in E^*\right\}.$$

Theorem 2.3.18 (Cramèr's Theorem)**.** *Let $(X_N)_{N\in\mathbb{N}}$ be a sequence of i.i.d. random variables taking values in a Banach space E, with common law μ. The sequence $\left(\overline{X}_N\right)_{N\in\mathbb{N}}$ of empirical means satisfies a weak large deviations principle on E, of speed N and with rate function Λ_μ^*.*

If $E = \mathbb{R}^d$ and D_{Λ_μ} contains a neighbourhood of zero we get a full LDP for $(X_N)_{N\in\mathbb{N}}$ and Λ_μ^* is a good, convex rate function.

Next we turn to the empirical measures of the X_N, whose large deviations behaviour is described by Sanov's theorem. Recall [100] that for a Polish space E the set $\mathfrak{M}_1(E)$ of probability measures on E, equipped with the weak topology, is itself a Polish space. In particular the weak topology on $\mathfrak{M}_+(E)$ is compatible with the complete separable metric β given for $\mu, \nu \in \mathfrak{M}_+(E)$ by

$$\beta(\mu, \nu) = \sup\left\{\int f \mathrm{d}\mu - \int f \mathrm{d}\nu \colon \|f\|_L + \|f\|_\infty\right\}$$

where $\|\cdot\|_L$, $\|\cdot\|_\infty$ denote the Lipschitz and supremum norms respectively.

For probability measures μ, ν on E we define their *relative entropy* (or *Kullback-Leibler divergence*) by [48]

$$H\left(\nu|\mu\right) = \begin{cases} \int \log\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) \mathrm{d}\nu & \text{if } \nu \ll \mu \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 2.3.19 (Sanov's Theorem)**.** *Let $(X_N)_{N\in\mathbb{N}}$ be i.i.d. random variables taking values in a Polish space E, of common law μ. Then the sequence of empirical* *measures* $\left(L_N^{(X)}\right)_{N\in\mathbb{N}}$ *(considered as random elements of* $\mathfrak{M}_1(E)$*) satisfies the LDP of speed* N *with good convex rate function* $H(\cdot|\mu)$*.*

Finally we discuss some sample path versions of Cramér's and Sanov's theorems. The path version of Cramér's theorem in $\mathbb{R}^d$ was established by Varadhan [120]. In the form we are stating it, the following result is due to Mogulskii [75]. Of course, by Proposition 2.3.10, the results presented below apply to closed subsets of $\mathbb{R}^d$. Denote by $L^\infty[0,1]$ the space of bounded $\mathbb{R}^d$-valued functions on the unit interval $[0,1]$, equipped with the supremum norm.

Theorem 2.3.20. *Let* $(X_N)_{N\in\mathbb{N}}$ *be a sequence of i.i.d.* $\mathbb{R}^d$*-valued random variables, of common law* ν *such that the logarithmic moment generating function of* ν *is finite on a neighbourhood of zero. Define, for* $N\in\mathbb{N}$*, the* $L^\infty[0,1]$*-valued random variable* Z_N *by*

$$Z_N(t) = \frac{1}{N}\sum_{j=1}^{\lfloor nt\rfloor} X_j$$

and denote its law by μ_N*. The sequence* $(\mu_N)_{N\in\mathbb{N}}$ *satisfies an LDP in* $L^\infty[0,1]$*, of speed* N *and with good rate function* I *defined by*

$$I(\phi) = \begin{cases} \int_0^1 \Lambda_\nu^*\left(\dot\phi(t)\right)\,\mathrm{d}t & \text{if } \phi\in\mathcal{A}_0 \\ +\infty & \text{otherwise} \end{cases}$$

where $\mathcal{A}_0$ *denotes the set of absolutely continuous elements* ϕ *of* $L^\infty[0,1]$ *such that* $\phi(0)=0$*.*

The following sample path version of Sanov's theorem was established by Dembo–Zajic [36]. Their result extends beyond the setting in which we state it here. In chapter 3 we will prove a joint pathwise Sanov-Cramér theorem for i.i.d $\mathbb{R}$-valued random variables whose moment generation function is finite on a neighbourhood of zero. We denote by D the space of càdlàg functions from $[0,1]$ to $\mathfrak{M}_+(\mathbb{R}^d)$, equipped with the norm

$$d_\infty(\boldsymbol{\mu},\boldsymbol{\nu}) = \sup\left\{\beta\big(\mu(t),\nu(t)\big) \colon t \in [0,1]\right\}.$$

Theorem 2.3.21. *Let $(X_N)_{N\in\mathbb{N}}$ be a sequence of i.i.d. $\mathbb{R}^d$ random variables of common law ν such that Λ_ν is finite on a neighbourhood of zero. Define further the D-valued random variable L_N by*

$$L_N(t) = \frac{1}{N}\sum_{j=1}^{\lfloor Nt\rfloor} \delta_{X_j}$$

and denote its law by μ_N. Then the sequence $(\mu_N)_{N\in\mathbb{N}}$ satisfies the LDP on D of speed N and with convex good rate function

$$I_\infty(\boldsymbol{\xi}) = \begin{cases} \int_0^1 \Lambda^*\left(\dot{\xi}(t)\right)\mathrm{d}t & \text{if } \nu \in \boldsymbol{A}_0 \\ +\infty & \text{otherwise.}\end{cases}$$

Here $\boldsymbol{A}_0$ denotes the set of maps $\boldsymbol{\xi}\colon [0,1] \longrightarrow \mathfrak{M}_+(E)$ that are absolutely continuous with respect to the total variation norm, have $\boldsymbol{\xi}(t) - \boldsymbol{\xi}(s)(E) \in \mathfrak{M}_{t-s}(E)$ and such that the limit

$$\dot{\boldsymbol{\xi}}(t) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\,\left(\xi\big(t+\epsilon\big) - \xi\big(t\big)\right)$$

exists in the weak topology for Lebesgue-almost all $t \in [0,1]$. The function Λ^* is defined by

$$\Lambda^*(\xi) = \sup\left\{\int f\mathrm{d}\xi - \log\int e^f(y)\nu(\mathrm{d}y) \colon f \in \mathcal{C}_b(E)\right\}.$$

2.3.4 Projective Limits

Because it avoids topological and analytical difficulties it is often easier to establish large deviations results on finite-dimensional spaces. Projective limits allow one to build up 'large' spaces from finite-dimensional building blocks. The Dawson–Gärtner theorem uses these projective limits to lift large deviations results from such subspaces to the larger space. We will use projective limits as a key ingredient in the proof of our joint Cramèr–Sanov theorem for i.i.d. random variables (Chapter 3).

In order to define projective limits we first need some topological preliminaries.

Definition 2.3.22. A partially ordered set $(J, \preceq)$ is said to be *right-filtering* if for any $j, k \in J$ there exists $l \in J$ such that both $j \preceq l$ and $k \preceq l$. A *projective system* is a collection of Hausdorff topological spaces $\{Y_j : j \in J\}$ indexed by a right-filtering set $(J, \preceq)$, together with continuous maps $p_{jk} \colon Y_j \longrightarrow Y_k$ for each $j, k \in J$ such that whenever $j \preceq k \preceq l$ we have $p_{jl} = p_{jk} \circ p_{kl}$.

It follows directly form the definition that for each $j \in J$ the map p_{jj} must be the identity map on Y_j.

Definition 2.3.23. The *projective (or inverse) limit* of such a projective system $(Y_j, p_{jk}; j, k \in J)$ is the topological subspace of the product space $\prod_{j \in J} Y_j$ consisting of all elements $y = (y_j)_{j \in J}$ with the property that $y_j = p_{jk}(y_k)$ whenever $j \preceq k$. We denote the projective limit by $X = \varprojlim Y_j$ and by $p_j \colon X \longrightarrow Y_j$ the restriction to X of the canonical projection from the product space.

Theorem 2.3.24 (Dawson–Gärtner). *Let $(\mu_N)_{N \in \mathbb{N}}$ be a sequence of probability measures on X. If for each $j \in J$ the sequence of probability measures $(\mu_N \circ p_j^{-1})_{N \in \mathbb{N}}$ satisfies the LDP on Y_j with good rate function I_j, then $(\mu_N)_{N \in \mathbb{N}}$ satisfies the LDP of the same speed and with good rate function $I \colon X \longrightarrow [0, \infty]$ given by*

$$I(\boldsymbol{y}) = \sup \{ I_j (p_j (\boldsymbol{y})) : j \in J \} .$$

2.4 Reflected BM and Queuing Theory

In this section we introduce *reflected Brownian motion* (RBM) in a polyhedral domain. There are two cases to consider: RBM in a general domain, driven by a standard Brownian motion and RBM in an orthant driven by a Brownian motion with general (possibly singular) covariance. We follow HARRISON–WILIAMS [53] and WILLIAMS [130] for the former and HARRISON–REIMAN [52] for the latter case.

Since the main motivation and many examples come from the study of queueing networks we also review some basic notions from queueing theory, based on the presentation in O'CONNELL–YOR [89].

2.4.1 RBM in a Polyhedral Domain

Let us first discuss RBM in a general domain. In the HARRISON–WILLIAMS setting the *polyhedral domain* $G \subseteq \mathbb{R}^d$ in which the process runs is the intersection of $k \geq d$ half-spaces. More precisely let $n_1, \ldots, n_k \in \mathbb{R}^d$ be unit vectors and $b \in \mathbb{R}^k$ then the domain G

$$G = \bigcap_{j=1}^{k} G_j := \bigcap_{j=1}^{k} \left\{ x \in \mathbb{R}^d \colon n_j \cdot x_j \geq b_j \right\}.$$

We assume that each of the *faces*

$$F_j = \left\{ x \in \overline{G} \colon n_j \cdot x = b_j \right\}$$

has dimension $d-1$. In general G may be bounded or unbounded, but we assume that $\{n_1, \ldots, n_k\}$ spans $\mathbb{R}^d$, which means that no line can lie entirely within $\overline{G}$.

The reflections are defined by vectors $q_1, \ldots, q_k \in \mathbb{R}^d$ such that $q_j \cdot n_j = 0$ for all j. We denote by N and Q the $k \times d$ matrices whose j^{th} rows are n_j and q_j respectively. The requirement that $\{n_1, \ldots, n_k\}$ spans all of $\mathbb{R}^d$ is equivalent to existence of an invertible $d \times d$ submatrix $\overline{N}$ of N.

Informally, reflected Brownian motion ω in G may be described as follows: inside the domain G the process ω behaves like a standard Brownian motion with drift $-\mu \in \mathbb{R}^d$, at the boundary it receives a singular drift pointing towards the interior – in direction $v_j := q_j + n_j$ whenever it hits the face F_j – and it almost surely never hits any point in the intersection of two or more faces.

In general, such a process may not exist. The boundary of the state space is not smooth, and the directions of reflection are discontinuous across non-smooth parts of the boundary, so this does not fit within the Stroock–Varadhan theory [115] of multidimensional diffusions. WILLIAMS [130] showed that if the input data satisfy the *skew-symmetry condition*

$$n_j \cdot q_r + n_r \cdot q_j = 0 \qquad \forall j, r \in \underline{k} \tag{2.4.1}$$

then there exists a reflected Brownian motion which can be defined as the unique solution to a submartingale problem. It was also shown in [130] that, under the skew-symmetry condition, reflected Brownian motion has an an invariant measure in product form:

Theorem 2.4.2. *Suppose that the skew-symmetry condition (5.1.1) holds, then RBM corresponding to (N, Q, μ, b) has a unique invariant measure whose density with respect to Lebesgue measure is given by*

$$p(x) = \exp\left\{2\gamma(\mu) \cdot x\right\} \tag{2.4.3}$$

where $\gamma(\mu)$ is defined as follows. By assumption, N has an invertible $d \times d$ submatrix $\overline{N}$. Denote the corresponding submatrix of Q by $\overline{Q}$, then

$$\gamma(\mu) = \left(I - \overline{N}^{-1}\overline{Q}\right)^{-1} \mu. \tag{2.4.4}$$

The fact that $\gamma(\mu)$ is independent of the choice of submatrix follows from the skew-symmetry condition. Harrison–Williams [53] also consider reflected Brownian motion in a smooth domain and establish similar results in that setting.

The fact that $\gamma(\mu)$ is independent of the choice of submatrix follows from the skew-symmetry condition. Harrison–Williams [53] also consider reflected Brownian motion in a smooth domain and establish similar results in that setting.

2.4.2 RBM in an Orthant

We now turn to reflected Brownian motion in the orthant $S = (0, \infty)^d$, driven by a general-covariance Brownian motion. Our definition almost exactly mirrors that of Harrison–Reiman [52]. However, we have changed the sign of the reflection matrix Q to make it compatible with the Harrison–Williams setup. This will be useful when we consider our generalised version in Chapter 5.

Let $d \in \mathbb{N}$ and B be a d-dimensional Brownian motion with drift μ and covariance matrix $A = \sigma\sigma^T$, started inside S. That is, there exists a k-dimensional standard Brownian motion β and a $k \times d$ matrix σ with unit rows such that $B(t) = \sigma\beta(t) - \mu t$. Let Q be a $d \times d$ matrix with non-negative entries and zeroes on the diagonals. Harrison–Reiman prove that there exists a unique pair of continuous $\mathbb{R}^d$-valued processes (Y, Z) with

$$Z(t) = B(t) + Y(t)(I + Q)$$

and such that

(i) $Z(t) \in S$ for all $t \geq 0$

(ii) for each $j \in \underline{d}$ the real-valued process Y_j is continuous, non-decreasing and such that $Y_j(0) = 0$

(iii) Y_j only increases at such times t where $Z_j(t) = 0$.

The process Z is called *reflected Brownian motion* in the orthant S with respect to the matrix Q, driven by B.

If $k = d$ and σ is an invertible matrix then it is easy to see that the process $\sigma^{-1}(Z)$ is a reflected Brownian motion in the polyhedral domain $\sigma^{-1}(S)$ in the sense of HARRISON–WILLIAMS.

In chapter 5 we will introduce generalisations of both these processes, replacing the singular drift by a continuous one that depends on how far the process fails to be in the relevant domain. We will then show that the same skew-symmetry condition still yields an invariant measure in product form.

2.4.3 Queuing Networks

We present here some basic notions and results from queueing theory, based on the presentation in [89], where analogues of Burke's theorem to the so-called *generalised Brownian queue* were used to compute the free energy density of a certain directed polymer in a random medium. See also MORIARTY–O'CONNELL [76]. Connections between tandem queues and directed percolation were investigated by O'CONNELL [85].

We will describe the M/M/1 queue, the Brownian queue and the generalised Brownian queue, stating for each case the relevant version of Burke's theorem. An introduction to queueing theory may be found in the book by KELLY [59], see also [4, 28, 99].

The classical *M/M/1* queue can be viewed as follows: the *arrivals* follow a Poisson process with parameter λ. A single server serves customers at the front of the queue, one at a time, where service times are distributed according to the exponential distribution with parameter $\mu > \lambda$. The M/M/1 queue is an example of a *birth and death process* on $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with birth rate λ and death rate μ.

More formally let A, S be independent Poisson processes on $\mathbb{R}$ with intensities

λ, μ respectively. The process Q defined by

$$Q(t) = \sup\{A(t) - A(s) - S(t) + S(s), 0\colon -\infty \leq s \leq t\}$$

is called the *queue-length process* of the M/M/1 queue. We refer to A and S as the *arivals* and *service process* respectively. The *departure process* D is defined by requiring that for $s < t$ we have

$$D(t) - D(s) = A(t) - A(s) + Q(s) - Q(t).$$

Burke's theorem [29] states that

1. the departures process D is itself a Poisson process with intensity λ
2. $\{D(t) - D(s)\colon s \leq t\}$ is independent of $\{Q(s)\colon s \geq t\}$ for any fixed $t \in \mathbb{R}$.

By letting the parameters λ, μ tend to infinity in the right way (which corresponds to considering a *heavy-traffic limit*), one can obtain a continous-time, *Brownian version* of Burke's theorem.

Following [89] we define a real-valued process $B = (B(t)\colon t \in \mathbb{R})$ indexed by the reals a *standard Brownian motion indexed by* $\mathbb{R}$ if $B(0) = 0$ and the two processes $(B(t)\colon t \geq 0)$ and $(B(-t)\colon t \geq 0)$ are two independent standard Brownian motions. Let now B, C be two such standard Brownian motions indexed by $\mathbb{R}$, fix $m > 0$ and define the *queue-length* and *departures* processes q, d by

$$q(t) = \sup\{B(t) - B(s) + C(t) - C(s) + m(s-t)\colon -\infty < s \leq t\} \tag{2.4.5}$$

$$d(t) = B(t) + q(0) - q(t). \tag{2.4.6}$$

The process d is also referred to as the *output* process. The system (B, C, q, d) is called the *Brownian queue*.

The Brownian analogue of Burke's theorem [54] is as follows:

1. d is a standard Brownian motion indexed by $\mathbb{R}$

2. for each $t \in \mathbb{R}$, $\{d(s), s \leq t\}$ is independent of $\{q(s) \colon s \geq t\}$.

The *generalised Brownian queue* is obtained from the above by replacing the supremum in (2.4.5) by $\log \int \exp$. More precisely, let once more B, C bet two independent standard Brownian motions indexed by $\mathbb{R}$ and $m > 0$. For $t \in \mathbb{R}$ we define

$$r(t) = \log \int_{-\infty}^{t} \exp\{B(t) - B(s) + C(t) - C(s) + m(s-t)\}\,\mathrm{d}s$$
$$f(t) = B(t) + r(0) - r(t).$$

The relevant version of Burke's theorem was shown in [89] to follow from results of Matsumoto–Yor [70], and states that f is a standard Brownian motion indexed by $\mathbb{R}$ and that for each t, the values of f up to t are independent from those of r after t.

Chapter 3

Large Deviations for Non-Crossing Partitions

3.1 Introduction

In this chapter we study the block structure of a *non-crossing partition* chosen uniformly at random. Recall from Section 2.1 that any partition π of the set $\underline{n} = \{1, \ldots, n\}$ can be represented on the circle by marking the points $1, \ldots, n$ and forming the convex hulls of the representatives of each block. The partition is then said to be *non-crossing* if none of the hulls intersect.

We study the empirical measure defined by the blocks of a uniformly random non-crossing partition π of $\underline{n}$. That is, if π has r blocks of sizes $B_1, \ldots, B_r$ we consider the random probability measure on $\mathbb{N}$ defined by

$$\lambda_n = \frac{1}{r} \sum_{j=1}^{r} \delta_{B_j}. \tag{3.1.1}$$

We will prove that the sequence $\big(\lambda_n\big)_{n \in \mathbb{N}}$ satisfies a large deviations principle of speed n on the space $\mathfrak{M}_1(\mathbb{N})$ of probability measures on the natural numbers.

This result is obtained via a construction of a uniformly random non-crossing partition by suitably conditioned independent geometric random variables. As a stepping-stone we establish a joint large deviations principle for the process versions of empirical mean and measure of that independent sequence.

A main application of the large deviations result comes from free probability theory. Recall that the free cumulants characterise the underlying probability distribution. However, obtaining the density involves locally inverting an analytic function which may not lead to a closed-form expression. In such a situation one would still hope to deduce some properties of the underlying probability measure, for example about its support.

The free analogue of the moment-cumulant formula expresses the moments of a non-commutative random variable in terms of its free cumulants. More precisely the moments can be written as the expectation of an exponential functional (defined in terms of the free cumulants) of a non-crossing partition, chosen uniformly at random. Knowing the large deviations behaviour of the latter allows us to apply Varadhan's lemma to describe the asymptotic behaviour of the moments. This in turn yields the maximum of the support of the underlying distribution in terms of the free cumulants.

3.2 Process Level Large Deviations

Let $(X_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of geometric random variables with parameter $\frac{1}{2}$ and denote their common law by $\mathfrak{G}_2$. We define processes $\boldsymbol{S}_n$, $\boldsymbol{L}_n$, indexed by the unit interval and taking values in the space of real numbers and positive finite measures on $\mathbb{N}$ respectively by

$$\boldsymbol{S}_n(t) = \frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor} X_j + \left(s - \frac{\lfloor ns\rfloor}{n}\right) X_{\lfloor ns\rfloor+1} \tag{3.2.1}$$

$$\boldsymbol{L}_n(t) = \frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor} \delta_{X_j} + \left(s - \frac{\lfloor ns\rfloor}{n}\right) \delta_{X_{\lfloor ns\rfloor+1}}. \tag{3.2.2}$$

In this section we prove a large deviations principle for the pair $(\boldsymbol{S}_n, \boldsymbol{L}_n)$. We start by proving a joint LDP for the pair of end-points $(\boldsymbol{S}_n(1), \boldsymbol{L}_n(1))$ via a projective limit argument. We then adapt arguments from Dembo–Zajic [36] to obtain the path-wise result.

Remark 3.2.3. By the sample-path version of Sanov's theorem we already have an LDP for $\boldsymbol{S}_n$ and $\boldsymbol{L}_n$ separately. The reason for obtaining this joint large deviations principle is that for our main large deviations result we need to use the mean as well as the empirical measure but the map $\mu \longmapsto m_1(\mu)$ is not continuous in the weak topology. An alternative would have been a priori to strengthen the topology on $\mathfrak{M}_+(\mathbb{N})$ to the *Monge–Kantorovich topology*, the coarsest topology that makes the map m_1 continuous and is finer than the weak topology. However results by Schied [102] show that in this topology Sanov's theorem only holds for distributions which possess all exponential moments. This does not hold for $\mathfrak{G}_2$ since

$$\int_R e^{tx}\, \mathfrak{G}_2(\mathrm{d}x) = +\infty$$

for all $t \geq \log 2$.

3.2.1 Joint Sanov and Cramér

Denote by $\mathfrak{M}_+(\mathbb{N})$ the space of finite measures on $\mathbb{N}$ and let $\mathfrak{M}_1(\mathbb{N})$ be the subset of probability measures. Recall that $\mathfrak{M}_+(\mathbb{N})$ is equipped with the topology of weak convergence, induced by the complete separable metric β given for $\mu, \nu \in \mathfrak{M}_+(\mathbb{N})$ by

$$\beta(\mu, \nu) = \sup\left\{ \left| \int f \mathrm{d}\mu - \int f \mathrm{d}\nu \right| : \|f\|_L + \|f\|_\infty \leq 1 \right\} \tag{3.2.4}$$

where $\|\cdot\|_L$ denotes the Lipschitz norm. So $\mathfrak{M}_+(\mathbb{N})$ is a Polish space, and so is $\mathfrak{M}_1(\mathbb{N})$ when equipped with the subspace topology.

Our goal here is to establish a joint large deviations principle on $\mathcal{Y} := \mathbb{R}\times\mathfrak{M}_+(\mathbb{N})$ (equipped with the product topology) for the empirical mean and measure of the X_n, i.e. for (S_n, L_n). where $S_n := \boldsymbol{S}_n(1) \in \mathbb{R}$ and $L_n := \boldsymbol{L}_n(1) \in \mathfrak{M}_1(\mathbb{N})$. By Cramér's theorem and Sanov's theorem respectively, the laws of S_n, L_n already satisfy a large deviations principle on $\mathbb{R}$ and $\mathfrak{M}_1(\mathbb{N})$ individually. The point here is to show that this also holds for the pair. Recall that $m_1(\mu)$ denotes the mean of a probability measure μ.

Proposition 3.2.5. *Let η_n denote the law of (S_n, L_n). Then $(\eta_n)_{n\in\mathbb{N}}$ satisfies a large deviations principle in $\mathcal{Y}$ with good rate function I_1 given by*

$$I_1(x,p) = \begin{cases} H(p|\mathfrak{G}_2) & \text{if } p \in \mathfrak{M}_1(\mathbb{N}) \text{ and } m_1(p) = x \\ +\infty & \text{otherwise} \end{cases} \tag{3.2.6}$$

where $H(\cdot|\cdot)$ denotes the relative entropy *of two probability measures, i.e.,*

$$H(\nu|\mathfrak{G}_2) = \sum_{m=1}^{\infty} \nu_m \log\left(\frac{\nu_m}{2^{-m}}\right) = m_1(\nu)\log(2) - H(\nu)$$

and $H(p) = -\sum_m p_m \log(p_m)$ is the entropy *of a probability measure p.*

Proof. The weak topology on $\mathfrak{M}_1(\mathbb{N})$ is induced by the dual action of the space $\mathcal{C}_b(\mathbb{N})$ of bounded continuous functions on $\mathbb{N}$. Fix a finite collection $f_1, \dots, f_d \in \mathcal{C}_b(\mathbb{N})$. The random variables $\big(X_n, f_1(X_n), \dots, f_d(X_n)\big) \in \mathbb{R}^{d+1}$ are independent and identically distributed, so by Cramér's theorem 2.3.18 their laws satisfy a large deviations principle on $\mathbb{R}^{d+1}$ with good convex rate function given by

$$\Lambda^*_{f_1,\dots,f_d}(x_1,\dots,x_{d+1}) = \sup\left\{\sum_{j=1}^{d+1} \lambda_j x_j - \Lambda_{f_1,\dots,f_d}(\lambda_1,\dots,\lambda_{d+1}) : \lambda \in \mathbb{R}^{d+1}\right\}$$

where $\Lambda_{f_1,\dots,f_d}$ is the logarithmic moment generating function of the random variable $\big(X_1, f_1(X_1), \dots, f_d(X_1)\big)$, that is

$$\Lambda_{f_1,\dots,f_d}(\lambda_1, \dots, \lambda_{d+1}) = \log \mathbb{E} e^{\lambda_1 X_1 + \sum_{j=1}^{d} \lambda_{j+1} f_j(X_1)}$$

The idea is now to take a projective limit approach. We first construct a suitable projective limit space in which $\mathbb{R} \times \mathfrak{M}_+(\mathbb{N})$ can be embedded. The proposition will follow from an application of the Dawson–Gärtner theorem (Theorem 2.3.24).

Denote $\mathcal{W} = \mathcal{C}_b(\mathbb{N})$ and let $\mathcal{W}'$ be its algebraic dual, equipped with the $\tau(\mathcal{W}', \mathcal{W})$-topology, that is the weakest topology making the maps $\mathcal{W}' \ni f \longmapsto f(w) \in \mathbb{R}$ continuous for all $w \in \mathcal{W}$. Let further J be the set of finite subspaces of $\mathcal{W}$, partially ordered by inclusion. For each $V \in J$ define $\mathcal{Y}_V = \mathbb{R} \times V'$ and equip it with the $\tau(\mathbb{R} \times V', \mathbb{R} \times V)$-topology. Defining now projection maps $p_{U,V}$ for each $U \subseteq V$ by

$$\begin{aligned} p_{U,V} \colon \mathcal{Y}_V &\longrightarrow \mathcal{Y}_U \\ p_{U,V}(x, f) &= (x, f|_U)\,, \end{aligned}$$

we obtain a projective system $(\mathcal{Y}_V, p_{U,V} \colon U \subseteq V \in J)$. Denote by $\widetilde{\mathcal{X}}$ its projective limit, equipped with the subspace topology from the product topology. Let further $\mathcal{X} = \mathbb{R} \times \mathcal{W}'$ and define $\Phi \colon \mathcal{X} \longrightarrow \widetilde{\mathcal{X}}$ by

$$\Phi(x, f) = ((x, f|_V) \colon V \in J)\,.$$

This is clearly a bijection. Using the definition of the weak topology in terms of open balls, as in Chapter 8 of Bollobás [25], it is clear that Φ is actually a homeomorphism.

Next we embed $\mathbb{R} \times \mathfrak{M}_1(\mathbb{N})$ into $\mathcal{X}$: for $(x, \mu) \in \mathbb{R} \times \mathfrak{M}_1(\mathbb{N})$ let

$$\Psi(x, \mu) = \left(x, \left[h \longmapsto \int h \, \mathrm{d}\mu \right] \right) \in \mathcal{X}.$$

Then Ψ is a homeomorphism onto its image, which we denote by $\mathcal{E}$. Let $\widetilde{\eta}_n$ be the image measure of η_n under Ψ. By the Dawson–Gärtner theorem and the finite-dimensional large deviations principle mentioned above, these satisfy a large deviations principle on $\mathcal{X}$ with good rate function I_Ψ given by

$$I_\Psi(x, f) = \sup \left\{ \Lambda^*_{\lambda_1, \dots \lambda_d} (x, f(\lambda_1), \dots, f(\lambda_d)) : \lambda_1, \dots \lambda_d \in \mathcal{W} \right\}.$$

By Cramér's and Sanov's theorem respectively we have exponential tightness for the sequences $(S_n)_{n \in \mathbb{N}}$ and $(L_n)_{n \in \mathbb{N}}$ separately. Therefore the sequence of *pairs* $((S_n, L_n))_{n \in \mathbb{N}}$ is exponentially tight in $\mathbb{R} \times \mathfrak{M}_1(\mathbb{N})$. The inverse contraction principle (Theorem 2.3.14) now yields the desired LDP for (S_n, L_n) with the good rate function $I_1 = I_\Psi \circ \Psi$. That is,

$$I_1(x, \mu) = \sup_{f_1, \dots, f_d \in \mathcal{C}_b(\mathbb{N})} \Lambda^*_{f_1, \dots, f_d} \left(x, \int f_1 \, \mathrm{d}\mu, \dots, \int f_d \, \mathrm{d}\mu \right).$$

It remains to show that I_1 is actually of the form (3.2.6). Suppose first that we have $m_1(\mu) = x$.

Fix $f_1, \dots, f_d \in \mathcal{C}_b(\mathbb{N})$, let $(\lambda_1, \dots, \lambda_{d+1}) \in \mathbb{R}^{d+1}$ and define $\phi(y) = \lambda_1 y + \sum_{j=1}^d \lambda_{j+1} f_j(y)$. By Jensen's inequality,

$$\begin{aligned} \log \mathbb{E}\left[e^{\phi(X_1)} \right] &= \log \int e^{\phi(y)} \frac{\mathrm{d}\mathfrak{G}_2}{\mathrm{d}\mu}(y) \, \mu(\mathrm{d}y) \geq \int \left[\phi(y) - \log \left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{G}_2}(y) \right) \right] \mu(\mathrm{d}y) \\ &= \int \phi \, \mathrm{d}\mu - H(\mu | \mathfrak{G}_2) = \lambda_1 x + \sum_{j=1}^d \lambda_{j+1} \int f_j \, \mathrm{d}\mu - H(\mu | \mathfrak{G}_2). \end{aligned}$$

So $\Lambda^*_{f_1, \dots, f_d} \left(x, \int f_1 \, \mathrm{d}\mu, \dots, f_1 \, \mathrm{d}\mu \right) \leq H(\mu | \mathfrak{G}_2)$ and therefore $I_1(x, \mu) \leq H(\mu | \mathfrak{G}_2)$

whenever $m_1(\mu) = x$.

If μ is a Dirac mass then $\mu = \delta_x$ and $x \in \mathbb{N}$, by the assumption that $m_1(\mu) = x$. So $H(\mu|\mathfrak{G}_2) = x \log 2$. On the other hand, choosing $d = 1$ and $f_1(y) = \mathbf{1}_{y<x}$ in the supremum definition of I_1 we obtain the inequality $I_1(x, \mu) \geq x \log 2$. Suppose now that μ is not a Dirac mass. Define $e_j \in \mathcal{C}_b(\mathbb{N})$ by $e_j(m) = \delta_{jm}$. Write $\operatorname{spt}(\mu) = \{n_k \colon k \in J\}$. Then,

$$\begin{aligned} I_1(x,\mu) &\geq \sup \left\{ \Lambda^*_{e_{n_1},\ldots,e_{n_d}}(x, \mu_{n_1}, \ldots, \mu_{n_d}) : d \in J \right\} \\ &= \sup_{\lambda \in (-\infty, \log(2)) \times \mathbb{R}^d} \left\{ \lambda_1 x + \sum_{j=1}^{d} \lambda_{j+1} \mu_{n_j} - \log \mathbb{E}\left[e^{\lambda_1 X_1 + \sum_{j=1}^d \lambda_{j+1} e_{n_j}(X_1)} \right] : \right\}. \end{aligned}$$

Fix $d \in J$, and let $g(\lambda)$ denote the function inside the supremum. The effective domain of $\Lambda_{e_{n_1},\ldots,e_{n_d}}$ is $(-\infty, \log(2)) \times \mathbb{R}^d$. Because μ is not a Dirac mass the function $g(\lambda)$ goes to $-\infty$ whenever $|\lambda|$ tends to ∞. So the supremum of g is attained at some $\lambda_0 \in (-\infty, \log(2)) \times \mathbb{R}^d$. Then λ_0 is a local maximum for g, whence $\nabla g(\lambda_0) = 0$, or equivalently $\nabla \Lambda_{e_{n_1},\ldots,e_{n_d}}(\lambda_0) = (x, \mu_{n_1}, \ldots, \mu_{n_d})^T$. So we can define an exponential tilting ν_{λ_0} of μ by

$$\nu_{\lambda_0}(\mathrm{d}y) = e^{\lambda_1 y + \sum_{j=1}^d \lambda_{j+1} e_{n_j}(y) - \Lambda_{e_{n_1},\ldots,e_{n_d}}(\lambda_0)} \mu(\mathrm{d}y).$$

The probability measure ν_{λ_0} has mean x and satisfies $\int e_{n_j}\, \mathrm{d}\mu = \int e_{n_j}\, \mathrm{d}\nu_{\lambda_0}$ for all $j \in \{1, \ldots, d\}$. Moreover,

$$H(\nu_{\lambda_0}|\mathfrak{G}_2) = \lambda_1 x + \sum_{j=1}^{d} \lambda_{j+1} \mu_{n_j} - \Lambda_{e_{n_1},\ldots,e_{n_d}}(\lambda_0) \leq \Lambda^*_{e_{n_1},\ldots,e_{n_d}}(x, \mu_{n_1}, \ldots, \mu_{n_d})$$

and therefore,

$$
\begin{aligned}
I_1(x,\mu) &\geq \sup_{d\in J} \inf \left\{ H\left(\nu|\mathfrak{G}_2\right) : m_1(\nu) = x,\, \nu_{n_j} = \mu_{n_j} \forall j \in \{1,\ldots,d\} \right\} \\
&= H\left(\mu|\mathfrak{G}_2\right).
\end{aligned}
$$

Finally suppose that $m_1(\mu) \neq x$. To see that $I_1(x,\mu) = +\infty$ we introduce, for each $d \in \mathbb{N}$, the function $\xi_d \in \mathcal{C}_b(\mathbb{N})$ by

$$
\xi_d(x) = \begin{cases} x & \text{if } x \leq d \\ 0 & \text{otherwise.} \end{cases}
$$

For each $\epsilon \in \left(0, \frac{1}{2}\left|x - m_1(\mu)\right|\right)$ there exists D such that for all $d \geq D$ we have

$$
m_1(\mu) \leq \int \xi_d \,\mathrm{d}\mu + \epsilon.
$$

It follows that for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $d \geq D$,

$$
\begin{aligned}
I_1(x,\mu) &\geq \lambda_1 x + \lambda_2\left(m_1(\mu) - \epsilon\right) - \log \mathbb{E} e^{\lambda_1 X_1 + \lambda_2 \xi_d(X_1)} \\
&= \left(\lambda_1 + \lambda_2\right) x - \log \mathbb{E} e^{(\lambda_1+\lambda_2)X_1} + \lambda_2\left(m_1(\mu) - x - \epsilon\right).
\end{aligned}
$$

Taking the supremum over all λ_1 we obtain

$$
I_1(x,\mu) \geq \Lambda^*(x) + \lambda_2\left(m_1(\mu) - x - \epsilon\right)
$$

for any $\lambda_2 \in \mathbb{R}$. Since $\epsilon < m_1(\mu) - x$ by construction it follows that $I_1(\mu) = \infty$.

It now follows from Proposition 2.3.10 that the LDP also holds in the larger space $\mathcal{Y} = \mathbb{R} \times \mathfrak{M}_+(\mathbb{N})$, by setting $I_1(x,\mu) = \infty$ whenever μ is not a probability measure. □

3.2.2 The Sample-Path Result

Theorem 3.2.7. *Let ξ_n denote the law of $(\boldsymbol{S}_n, \boldsymbol{L}_n)$ on $\mathcal{C}([0,1];\mathcal{Y})$, the space of continuous functions from the unit interval to $\mathcal{Y}$. The sequence $(\xi_n)_{n\in\mathbb{N}}$ satisfies a large deviations principle on $\mathcal{C}([0,1];\mathcal{Y})$ with good rate function I_2 given by*

$$I_2(\boldsymbol{x}, \boldsymbol{p}) = \begin{cases} \int_0^1 H\left(\dot{\boldsymbol{p}}(s)|\mathfrak{G}_2\right) \mathrm{d}s & \text{if } (\boldsymbol{p}, \boldsymbol{x}) \in \mathcal{E} \\ +\infty & \text{otherwise} \end{cases} \tag{3.2.8}$$

where $\mathcal{E}$ is the space of elements $(\boldsymbol{m}, \boldsymbol{p})$ of absolutely continuous maps $[0,1] \longrightarrow \mathcal{Y}$ such that $(\boldsymbol{m}(0), \boldsymbol{p}(0)) = 0$, the map $s \longmapsto \boldsymbol{m}(s)$ is differentiable almost everywhere, $\boldsymbol{p}(t) - \boldsymbol{p}(s) \in \mathfrak{M}_{t-s}(\mathbb{N})$ and the limit

$$\dot{\boldsymbol{p}}_t = \lim_{\epsilon \to 0} \frac{\boldsymbol{p}_{t+\epsilon} - \boldsymbol{p}_t}{\epsilon}$$

exists in the weak topology for almost every $t \in [0,1]$ and has the property that $m_1(\dot{\boldsymbol{p}}(\cdot)) = \boldsymbol{m}(\cdot)$.

For $(\boldsymbol{L}_n(\cdot))$ on its own the analogous result can be found in Dembo–Zajic [36] and we will use a similar approach, using the joint large deviations principle for empirical mean and measure established above. We first prove exponential tightness for the pair of paths:

Lemma 3.2.9. *$((\boldsymbol{S}_n(\cdot), \boldsymbol{L}_n(\cdot)))_{n\in\mathbb{N}}$ is exponentially tight in $\mathcal{C}([0,1];\mathcal{Y})$.*

Proof. The topology on $\mathcal{Y}$ is induced by the metric d given by

$$d\left((x_1, \mu_1), (x_2, \mu_2)\right) = |x_1 - x_2| + \beta(\mu_1, \mu_2).$$

By Lemma A.2 in [36] we get exponential tightness for the laws ξ_n of $(\boldsymbol{S}_n, \boldsymbol{L}_n)$ if

(a) for each fixed $t \in \mathcal{Q} \cap [0,1]$ the sequence $((\boldsymbol{S}_n(t), \boldsymbol{L}_n(t)))_{n\in\mathbb{N}}$ is exponentially tight and

(b) for every $\rho > 0$,

$$\lim_{\delta\to 0} \sup_{n\in\mathbb{N}} \frac{1}{n} \log \xi_n \{f \colon w_f(\delta) \geq \rho\} = -\infty$$

where $w_f(\delta) = \sup_{|t-s|\leq\delta} d\big(f(t), f(s)\big)$ is the *modulus of continuity* of f.

Exponential tightness of $((\boldsymbol{S}_n(t), \boldsymbol{L}_n(t)))_{n\in\mathbb{N}}$ for every *fixed* $t \in \mathcal{Q} \cap [0,1]$ is a direct consequence of Proposition 3.2.5. Moreover, for $0 \leq s < t \leq 1$,

$$d\left((\boldsymbol{S}_n(t), \boldsymbol{L}_n(t)), (\boldsymbol{S}_n(s), \boldsymbol{L}_n(s))\right) \leq \frac{t-s}{n} \max_j X_j + \frac{t-s}{n}$$

where the maximum on the right-hand side runs over the (finite) set of j such that $\lfloor ns \rfloor \leq j \leq \lfloor nt \rfloor$. For any $\delta, \rho > 0$ and $n \in \mathbb{N}$ it follows therefore that

$$\begin{aligned}\frac{1}{n} \log \mathbb{P} &\left\{ \sup_{|t-s|<\delta} d\left((\boldsymbol{S}_n(t), \boldsymbol{L}_n(t)), (\boldsymbol{S}_n(s), \boldsymbol{L}_n(s))\right) \geq \rho \right\} \\ &\leq \frac{1}{n} \log \mathbb{P} \left\{ \frac{\delta}{n} \left(\max_{1\leq j\leq n} X_j + 1 \right) \geq \rho \right\} = -\left(\frac{n\rho}{\delta} - 1\right) \log 2 \leq -\left(\frac{\rho}{\delta} - 1\right) \log 2.\end{aligned}$$

The right-hand side diverges to $-\infty$ as $\delta \to 0$. So condition (b) also holds and $(\xi_n)_{n\in\mathbb{N}}$ is exponentially tight. □

Lemma 3.2.10. *For any fixed* $0 = t_0 < t_1, \ldots, < t_m \leq 1$ *the sequence* $(Z_n)_{n\in\mathbb{N}}$ *of random variables defined by*

$$Z_n = \left((\boldsymbol{S}_n(t_j) - \boldsymbol{S}_n(t_{j-1}), \boldsymbol{L}_n(t_j) - \boldsymbol{L}_n(t_{j-1}))\right)_{j=1}^m \in \mathcal{Y}^m$$

satisfies a large deviations principle in $\mathcal{Y}^m$ with good rate function given by

$$I_{t_1,\dots,t_m}\left((x_1,\mu_1),\dots,(x_m,\mu_m)\right) = \sum_{j=1}^{m} (t_j - t_{j-1})\, I_1\left(\frac{x_j}{t_j - t_{j-1}}, \frac{\mu_j}{t_j - t_{j-1}}\right).$$

Proof. Let n be large enough so that $nt_j < nt_{j+1} - 1$. A direct calculation yields, for any $f = (\lambda_j, g_j)_{j=1}^m \in (\mathcal{Y}^m)^*$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\, e^{nf(Z_n)} = \sum_{j=1}^{m} (t_j - t_{j-1})\, \Lambda_2\left(\frac{\lambda_j}{t_j - t_{j-1}}, \frac{g_j}{t_j - t_{j-1}}\right) =: \Lambda_3(f)$$

where $\Lambda_2(\lambda, g) = \log\, \mathbb{E}\left[\exp\left(\lambda X_1 + g(\delta_{X_1})\right)\right]$.

By Corollary 4.6.14 of [37] this implies that the laws of Z_n satisfy a large deviations principle on $\mathcal{E}$ with good rate function Λ_1^* given by

$$\begin{aligned}\Lambda_1^*\left((x_j,\mu_j)_{j=1}^m\right) &= \sup\left\{ f\left((x_j,\mu_j)_{j=1}^m - \Lambda_3(f) \colon f \in \mathcal{E}^*\right)\right\}\\ &= \sum_{j=1}^{d} (t_j - t_{j-1})\, \Lambda_2^*\left(\frac{x_j}{t_j - t_{j-1}}, \frac{\mu_j}{t_j - t_{j-1}}\right).\end{aligned}$$

Since I_1 is convex it follows from the results of Section 3.2.1 and Theorem 4.5.10(b) in [37] that $\Lambda_2^* = I_1$ and the lemma is proved. □

The proof of Theorem 3.2.7 now follows closely that of Theorem 1 of [36]. An application of the contraction principle to the map

$$(z_1,\dots,z_m) \longmapsto (z_1, z_1 + z_2, \dots, z_1 + \dots z_m)$$

yields the large deviations principle for the laws of $(\boldsymbol{S}_n(t_1), \boldsymbol{L}_n(t_1), \dots, \boldsymbol{S}_n(t_m), \boldsymbol{L}_n(t_m))$ with good convex rate function given by

$$\widehat{I}_{t_1,\dots,t_m}\left((x_1,\mu_1),\dots,(x_m,\mu_m)\right) = \sum_{j=1}^{m} (t_j - t_{j-1})\, I_1\left(\frac{x_j - x_{j-1}}{t_j - t_{j-1}}, \frac{\mu_j - \mu_{j-1}}{t_j - t_{j-1}}\right).$$

Applying the Dawson–Gärtner theorem as in the proof of Lemma 3 in [36] yields an LDP for the laws of the pair process $(\boldsymbol{S}_n, \boldsymbol{L}_n)$ on $\mathcal{C}([0,1];\mathcal{Y})$ with good rate function

$$I_2(\boldsymbol{x}, \boldsymbol{\mu}) = \sup_{t_1 < \ldots < t_m} \widehat{I}_{t_1,\ldots,t_m}\big(\boldsymbol{x}(t_1), \boldsymbol{\mu}(t_1), \ldots, \boldsymbol{x}(t_m), \boldsymbol{\mu}(t_m)\big).$$

Obviously $m_1(\boldsymbol{\mu}(t)) \neq \boldsymbol{x}(t)$ for some t implies $I_2(\boldsymbol{x}, \boldsymbol{\mu}) = \infty$. Lemma 4 in [36] then implies that I_2 is of the form (3.2.8). This completes the proof of Theorem 3.2.7. □

Finally let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ be *two* sequences of i.i.d. random variables of common law $\mathfrak{G}_2$ and define $\boldsymbol{L}_n^X$, $\boldsymbol{L}_n^Y$, $\boldsymbol{S}_n^X$, $\boldsymbol{S}_n^Y$ analogously to (3.2.1, 3.2.2). By Corollary 2.9 of Lynch–Sethuraman [65] we obtain the following

Corollary 3.2.11. *The sequence of the laws of* $(\boldsymbol{S}_n^X, \boldsymbol{L}_n^X, \boldsymbol{S}_n^Y, \boldsymbol{L}_n^Y)$ *satisfies a large deviations principle on* $\mathcal{C}([0,1],\mathcal{Y}^2)$ *with good rate function* I *where for* $(\boldsymbol{x},\boldsymbol{p},\boldsymbol{y},\boldsymbol{q}) \in \mathcal{Y}^2$,

$$I(\boldsymbol{x},\boldsymbol{p},\boldsymbol{y},\boldsymbol{q}) = \begin{cases} \int_0^1 \left[H\left(\dot{\boldsymbol{p}}(s)|\mathfrak{G}_2\right) + H\left(\dot{\boldsymbol{q}}(s)|\mathfrak{G}_2\right)\right] \mathrm{d}s & \text{if } (\boldsymbol{x},\boldsymbol{p}), (\boldsymbol{y},\boldsymbol{q}) \in \mathcal{E} \\ +\infty & \text{otherwise.} \end{cases}$$

3.3 Construction of the Uniform Measure on NC(n)

Let us now turn to the construction of the uniform measure on NC(n) using conditioned geometric random variables. Since the sets NC(n) and $\mathcal{P}(n)$ are finite, a uniform distribution exists. A Dyck path chosen uniformly at random is also referred to as *Bernoulli excursion*. We will study the descent structure of ω. Because of our bijection between $\mathcal{P}(n)$ and $\mathrm{NC}(n)$ this is equivalent to studying the blocks of a uniformly random element of $\mathrm{NC}(n)$.

We now construct a Bernoulli excursion using conditioned geometric random variables. For any $n \in \mathbb{N}$ let $w_n \colon \mathbb{N}^{2n} \longrightarrow \bigcup_{k\in\mathbb{N}} \widehat{\mathcal{P}}(k)$ (where $\widehat{\mathcal{P}}(k)$ is the set of *all* length $2k$ lattice paths on $\mathbb{Z}$, starting at zero and consisting of steps $(1,1)$ and $(1,-1)$) denote the map that reconstructs a path from a sequence of ascents and descents. That is, $b_n(x_1, y_1, \ldots, x_n, y_n)$ is the path described by x_1 upsteps, y_1 downsteps, then x_2 upsteps and so on, terminating with y_n downsteps.

Let X_n, Y_n be i.i.d. random variables with common law given by the geometric distribution with parameter $\frac{1}{2}$. We will view these as the subsequent ascents and descents of a simple random walk Σ on $\mathbb{R}$ starting at 0 with an upstep. Denote by

$$T_n := \sum_{j=1}^{n} (X_j + Y_j)$$

the combined length the first n up- and downsteps take in total and let $\widehat{\tau}_n$ be the number of descents completed after $2n$ steps of the simple random walker:

$$\widehat{\tau}_n = \max\{k \in \mathbb{N} \colon T_k \leq 2n\}.$$

We will later work with a renormalisation of $\widehat{\tau}_n$, namely $\tau_n = \frac{\widehat{\tau}_n}{2n}$. We denote by E_n the event that $w_{\tau_n}(X_1, Y_1, \ldots, X_{\tau_n}, Y_{\tau_n})$ is a Dyck path of semilength n:

$$E_n = \left\{ T_{\widehat{\tau}_n} = 2n,\ \sum_{j=1}^{\widehat{\tau}_n} X_j = \sum_{j=1}^{\widehat{\tau}_n} Y_j,\ \sum_{j=1}^{r} (X_j - Y_j) \geq 0 \ \forall\, j < \widehat{\tau}_n \right\}. \tag{3.3.1}$$

The following lemma is now straightforward to check.

Lemma 3.3.2. *Conditioned on E_n the distribution of $w_{\widehat{\tau}_n}(X_1, Y_1, \ldots, X_{\widehat{\tau}_n}, Y_{\widehat{\tau}_n})$ on*

$\mathcal{P}(n)$ is uniform. Hence, conditioned on E_n, the random measure λ_n defined by

$$\lambda_n = \frac{1}{\widehat{\tau}_n} \sum_{j=1}^{\widehat{\tau}_n} \delta_{Y_j} \tag{3.3.3}$$

is the empirical measure of the descents of a Bernoulli excursion or, equivalently, the block sizes of a uniformly random element of $NC(n)$.

3.4 Large Deviations for Non-Crossing Partitions

Recall that $\lambda_n = \frac{1}{\widehat{\tau}_n} \sum_{j=1}^{\tau_n} \delta_{Y_j}$ is the empirical measure of the blocks of a non-crossing partition picked uniformly at random. Define further $\sigma_n = m_1(\lambda_n) = \frac{1}{\widehat{\tau}_n} \sum_{j=1}^{\widehat{\tau}_n} Y_j$.

Let ν_n denote the law of $(\sigma_n, \lambda_n, \tau_n)$ on $\mathcal{Y} \times [0,1]$. The main result of this section is the following.

Theorem 3.4.1. *The sequence $(\sigma_n, \lambda_n, \tau_n)_{n\in\mathbb{N}}$ satisfies a large deviations principle in $\mathcal{Y} \times [0,1]$ with good convex rate function J given by*

$$J(m,\mu,t) = \begin{cases} \log 4 - \frac{1}{m} H(\mu) - \frac{1}{m}\log(m-1) + \log\left(1 - \frac{1}{m}\right) & \text{if } m_1(\mu) = m = \frac{1}{2t} \\ +\infty & \text{otherwise.} \end{cases} \tag{3.4.2}$$

It is straightforward to verify that $J(m,\mu,t) = 0$ if and only if $(m,\mu,t) = (2, \mathfrak{G}_2, \frac{1}{4})$. The following law of large numbers now follows immediately.

Corollary 3.4.3. *The empirical measure λ_n of the block structure of a uniformly randomly chosen non-crossing partition converges weakly almost surely to the geometric distribution of parameter $\frac{1}{2}$.*

We will first prove the upper bound, Proposition 3.4.11 and then the lower bound, Proposition 3.4.13. For both the following lemma is useful.

Lemma 3.4.4. *The logarithmic asymptotics of the probability of E_n are given by*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(E_n) = 0. \tag{3.4.5}$$

Proof. Writing E_n in terms of the simple random walk Σ that has ascents $X_1, X_2, \ldots$ and descents $Y_1, Y_2, \ldots$,

$$E_n = \big\{\Sigma_{2n} = 0,\, \Sigma_k > 0 \;\forall\, k < 2n,\, \Sigma_{2n+1} = +1\big\}.$$

Therefore, using the Markov property of Σ,

$$\begin{aligned}\mathbb{P}\big(E_n\big) &= \mathbb{P}\left\{\Sigma_{2n+1} = +1 \mid \Sigma_{2n} = 0\right\} \cdot \mathbb{P}\left\{\Sigma_{2n} = 0,\, \Sigma_k \geq 0 \;\forall\, k < 2n\right\} \\ &= \frac{1}{2} \cdot \frac{C_n}{4^n}\end{aligned}$$

because the second probability on the right is just that of running a simple random walk for $2n$ steps and obtaining a Dyck path. A direct computation using Stirling's formula [43, p.64] yields that $\frac{1}{n} \log C_n \longrightarrow 4$ as $n \to \infty$. Equation (3.4.5) follows. $\square$

For any path $x\colon [0,1] \longrightarrow \mathbb{R}$ with $x(0) = 0$ and $x(t) - x(s) \geq t - s$ for all $t > s \geq 0$ we let $\tau(x)$ be the right-inverse of x at 1, i.e.

$$\tau(x) = \inf\left\{s \in [0,1]\colon x(s) \geq 1\right\}.$$

If $\boldsymbol{p}(t) - \boldsymbol{p}(s)$ is a measure on $\mathbb{N}$ of mass $t - s$ it follows that $m_1\left(\boldsymbol{p}(t)\right) - m_1\left(\boldsymbol{p}(s)\right) \geq t - s$. So the map $\mathcal{E}^2 \longrightarrow [0,1]$ given by $(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{y}, \boldsymbol{q}) \longmapsto \tau(\boldsymbol{x} + \boldsymbol{y})$ is continuous.

3.4.1 The Upper Bound

We are now in a position to prove the large deviations upper bound. We will first give a bound via the process version and then show that this can be written in terms of the stated rate function

Lemma 3.4.6. *For every closed $F \subseteq \mathcal{Y} \times [0,1]$ we have*

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_n(F) \le -2 \inf \left\{ I(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{y}, \boldsymbol{q}) \colon (\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{y}, \boldsymbol{q}) \in \widehat{F} \right\} \tag{3.4.7}$$

where the (closed) subset $\widehat{F}$ of $\mathbb{E}^2$ is defined by

$$\widehat{F} = \left\{ (\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{y}, \boldsymbol{q}) \colon \left(\frac{1}{\tau} \boldsymbol{y}(\tau), \frac{1}{\tau} \boldsymbol{q}(\tau), \tau \right) \in F, \boldsymbol{x}(\tau) = \boldsymbol{y}(\tau), \ \ \boldsymbol{x}(s) \ge \boldsymbol{y}(s) \, \forall s \le \tau \right\}$$

and $\tau = \tau(\boldsymbol{x} + \boldsymbol{y})$.

Proof. Recall that $\lambda_n = \frac{1}{\tau_n} \boldsymbol{L}^Y_{2n}(\tau_n)$. Therefore,

$$\frac{1}{n} \log \nu_n(F) = \frac{1}{n} \log \mathbb{P} \left\{ \left(\frac{1}{\tau_n} \boldsymbol{S}^Y_{2n}(\tau_n), \frac{1}{\tau_n} \boldsymbol{L}^Y_{2n}(\tau_n), \tau_n \right) \in F; \ E_n \right\} - \frac{1}{n} \log \mathbb{P}(E_n).$$

By Lemma 3.4.4 the second term on the right-hand side converges to 0. Further, $\tau_n = \inf \left\{ \frac{k}{2n} \colon \frac{1}{2n}(X_j + Y_j) \ge 1 \right\}$, so that τ_n is the least integer multiple of $\frac{1}{2n}$ less than $\tau(\boldsymbol{L}^X_{2n} + \boldsymbol{L}^Y_{2n})$, with equality if and only if $\boldsymbol{S}^X_{2n}(\tau_n) + \boldsymbol{S}^Y_{2n}(\tau_n) = 1$. This certainly holds on E_n, so we can write the event E_n in terms of the $\boldsymbol{L}$, $\boldsymbol{S}$: for ease of notation we denote $\tau^{\boldsymbol{S}} := \tau(\boldsymbol{S}^X_{2n} + \boldsymbol{S}^Y_{2n})$. Then

$$\begin{aligned} E_n &= \left\{ \boldsymbol{S}^X_{2n}(\tau_n) = \boldsymbol{S}^Y_{2n}(\tau_n) = \frac{1}{2}, \, \boldsymbol{S}^X_{2n}(s) \ge \boldsymbol{S}^Y_{2n}(s) \ \forall \, s \le \tau_n \right\} \\ &= \left\{ \boldsymbol{S}^X_{2n}(\tau^{\boldsymbol{S}})) = \boldsymbol{S}^Y_{2n}(\tau^{\boldsymbol{S}}) = \frac{1}{2}, \, \boldsymbol{S}^X_{2n}(s) \ge \boldsymbol{S}^Y_{2n}(s) \ \forall \, s \le \tau^{\boldsymbol{S}}, \ \tau^{\boldsymbol{S}} = \tau_n \right\} \\ &\subseteq \left\{ \boldsymbol{S}^X_{2n}(\tau^{\boldsymbol{S}})) = \boldsymbol{S}^Y_{2n}(\tau^{\boldsymbol{S}}) = \frac{1}{2}, \, \boldsymbol{S}^X_{2n}(s) \ge \boldsymbol{S}^Y_{2n}(s) \ \forall \, s \le \tau^{\boldsymbol{S}} \right\}. \end{aligned}$$

Denote the last event by $\widetilde{E}_n$. It follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu_n(F) \le 2 \limsup_{n\to\infty} \frac{1}{2n} \log \mathbb{P} \left\{ \left(\frac{1}{\tau^{\boldsymbol{S}}} \boldsymbol{S}^Y_{2n}(\tau^{\boldsymbol{S}}), \frac{1}{\tau^{\boldsymbol{S}}} \boldsymbol{L}^Y_{2n}(\tau^{\boldsymbol{S}}), \tau^{\boldsymbol{S}} \right) \in F; \ \widetilde{E}_n \right\}.$$

Since $\tau^{\boldsymbol{S}}$ is a continuous function of $(\boldsymbol{S}_{2n}^X, \boldsymbol{L}_{2n}^X, \boldsymbol{S}_{2n}^Y, \boldsymbol{L}_{2n}^Y)$, the set on the right-hand side is closed in $\mathcal{E}^2$ and we can apply Corollary 3.2.11 to obtain (3.4.7). $\square$

We now investigate the right-hand side of (3.4.7). For any $(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{y}, \boldsymbol{q})$ define new paths $\widetilde{p}$ and $\widetilde{q}$ by

$$\widetilde{p}(s) = \begin{cases} \frac{s}{\tau(\boldsymbol{x}+\boldsymbol{y})}\boldsymbol{p}\left(\tau\left(\boldsymbol{x}+\boldsymbol{y}\right)\right) & \text{if } s \in [0, \tau\left(\boldsymbol{x}+\boldsymbol{y}\right)] \\ \boldsymbol{p}\left(\tau\left(\boldsymbol{x}+\boldsymbol{y}\right)\right) + \left(s - \tau\left(\boldsymbol{x}+\boldsymbol{y}\right)\right)\mathfrak{G}_2 & \text{if } s \in [\tau\left(\boldsymbol{x}+\boldsymbol{y}\right), 1] \end{cases} \tag{3.4.8}$$

and analogously $\widetilde{q}$, replacing $\boldsymbol{p}$ by $\boldsymbol{q}$. If further $\widetilde{x}(t) = m_1(\widetilde{p}(t))$ and $\widetilde{y}(t) = m_1(\widetilde{q}(t))$ for all t then $\tau\left(\boldsymbol{x}+\boldsymbol{y}\right) = \tau\left(\widetilde{x}+\widetilde{y}\right) =: \tau$. Also $(\widetilde{x}, \widetilde{p}, \widetilde{y}, \widetilde{q}) \in \widehat{F}$ and

$$I(\widetilde{x}, \widetilde{p}, \widetilde{y}, \widetilde{q}) = \tau\left(H\left(\frac{1}{\tau}\boldsymbol{p}(\tau)|\mathfrak{G}_2\right) + H\left(\frac{1}{\tau}\boldsymbol{q}(\tau)|\mathfrak{G}_2\right)\right).$$

Moreover, by convexity of $H(\cdot|\mathfrak{G}_2)$ (cf. [36], Lemma 4),

$$I(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{y}, \boldsymbol{q}) \geq \tau\left(H\left(\frac{1}{\tau}\boldsymbol{p}(\tau)|\mathfrak{G}_2\right) + H\left(\frac{1}{\tau}\boldsymbol{q}(\tau)|\mathfrak{G}_2\right)\right).$$

It is clear that $\frac{1}{\tau}\boldsymbol{p}(\tau)$, $\frac{1}{\tau}\boldsymbol{q}(\tau)$ are probability measures, and that for every pair of probability measures (p, q) such that $(m_1(p), p, \frac{1}{2m_1(p)}) \in F$ the corresponding straight-line path $\big((3.4.8)$ with $\tau(\boldsymbol{x}+\boldsymbol{y}) = \frac{1}{2m_1(p)}\big)$ lies in $\widehat{F}$. Therefore

$$\inf_{\widehat{F}} I = \inf\left\{\tau\left[H\left(p|\mathfrak{G}_2\right) + H\left(q|\mathfrak{G}_2\right)\right] : (m_1(q), q, \tau) \in F, m_1(p) = m_1(q) = \frac{1}{2\tau}\right\}.$$

On the other hand $H(q|\mathfrak{G}_2) = -H(q) + m_1(p)\log(2)$ and it is well-known that

$$\sup\left\{H(q)\colon m_1(q) = m\right\} = \Theta(m) := \log(m-1) - m\log\left(1 - \frac{1}{m}\right). \tag{3.4.9}$$

Hence,

$$\inf_{\widehat{F}} I = \inf \left\{ \log(2) - \tau H(p) - \tau \Theta\left(\frac{1}{2\tau}\right) : (m_1(p), p, \tau) \in F, m_1(p) = \frac{1}{2\tau} \right\}. \tag{3.4.10}$$

We have established the upper bound:

Proposition 3.4.11. *For every closed $F \subset \mathcal{Y} \times [0,1]$,*

$$\liminf_{m\to\infty} \frac{1}{m} \log \nu_m(F) \leq - \inf \{J(s,p,t) \colon (s,p,t) \in F\}. \tag{3.4.12}$$

3.4.2 The Lower Bound

We now turn to proving the lower bound. By the local nature of large deviations lower bounds, cf. (2.3.6), it is enough to prove the following.

Proposition 3.4.13. *Fix $(m,\mu,t) \in \mathcal{Y} \times [0,1]$ and $\rho_j > 0$ for $j \in \{1,2,3\}$ and let $G = (m-\rho_2, m+\rho_2) \times B(\mu,\rho_1) \times (t-\rho_3, t+\rho_3)$. Then*

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_n\big(G\big) \geq -J(m,\mu,t) \tag{3.4.14}$$

where $B(\mu, r)$ denotes the ball in $\mathfrak{M}_1$ of radius r, centred on μ with respect to β, the metric of (3.2.4) inducing weak topology.

Proof. We can assume that $m_1(\mu) = m = \frac{1}{2t}$ since otherwise $J(m,\mu,t) = \infty$ and (3.4.14) is trivial. From the definition of ν_n we have, as before,

$$\log\big(\nu_n(G)\big) = \log \mathbb{P}\{(\sigma_n, \lambda_n, \tau_n) \in G\} - \log \mathbb{P}(E_n).$$

Recall that $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(E_n) = 0$. Moreover $E = \bigcup_r E_{n,r}$ where

$$E_{n,r} = \left\{ \sum_{j=1}^{r} X_j = \sum_{j=1}^{r} Y_j,\ \sum_{j=1}^{k} X_j \geq \sum_{j=1}^{k} Y_j \ \forall\, k < r \ \ \tau_n = \frac{r}{2n} \right\}.$$

On $E_{n,r}$ we have $\frac{r}{2n} = \tau_n \in (t-\rho_3, t+\rho_3)$ and the condition $\sigma_n \in (m-\rho_2, m+\rho_2)$ is equivalent to $r \in \left(\frac{n}{m+\rho_2}, \frac{n}{m-\rho_2}\right)$. Therefore

$$\mathbb{P}\left\{(\lambda_n, \sigma_n, \tau_n) \in G; E_n\right\} = \sum_{r\in\mathcal{I}_n} \mathbb{P}\left\{ \frac{1}{r}\sum_{j=1}^{r} \delta_{Y_j} \in B(\mu,\rho_1);\ E_{n,r} \right\}$$

where $\mathcal{I}_n = \mathbb{N} \cap (2n(t-\rho_3), 2n(t+\rho_3)) \cap \left(\frac{n}{m+\rho_2}, \frac{n}{m-\rho_2}\right)$. Fix now $w > 0$ and let N_1 be large enough to have $N_1 w > \frac{2}{\rho_1}$. Then if $r \in \mathcal{I}_n^{(w)} = \mathcal{I}_n \cap (wn, \infty)$ and $n \geq N_1$,

$$\beta\left(\frac{1}{r}\sum_{j=1}^{r}\delta_{Y_j}, \frac{1}{r}\sum_{j=1}^{r-1}\delta_{Y_j} \right) = \frac{1}{r} < \frac{\rho_1}{2}.$$

Using independence of the X_j, Y_j and the fact that $\mathbb{P}(Z = a) = \mathbb{P}(Z > a)$ for any Z with law $\mathfrak{G}_2$,

$$\begin{aligned}
\mathbb{P}\left\{(\lambda_n, \sigma_n, \tau_n) \in G;\ E_n\right\} &\geq \frac{1}{4} \sum_{r\in\mathcal{I}_n^{(w)}} \mathbb{P}\left\{ \frac{1}{r}\sum_{j=1}^{r-1}\delta_{Y_j} \in B\left(\mu, \frac{\rho_1}{2}\right),\ E'_{n,r},\ \tau_n = \frac{r}{2n} \right\} \\
&\geq \frac{1}{4} \sum_{r\in\mathcal{I}_n^{(w)}} \mathbb{P}\left\{ \frac{1}{r}\sum_{j=1}^{r}\delta_{Y_j} \in B\left(\mu, \frac{\rho_1}{4}\right),\ E'_{n,r},\ \tau_n = \frac{r}{2n} \right\}
\end{aligned}$$

for $n \geq 2N_1$. Here,

$$E'_{n,r} = \left\{ \frac{1}{r}\sum_{j=1}^{r} X_j \geq n,\ \frac{1}{r}\sum_{j=1}^{r} Y_j \geq n,\ \sum_{j=1}^{a} X_j \geq \frac{1}{r}\sum_{j=1}^{a} Y_j \ \forall a < r, \tau_n = \frac{r}{2n} \right\}.$$

Recall that $\lambda_n = \frac{1}{\tau_n}\boldsymbol{L}^Y_{2n}(\tau_n)$, that $\tau_n - \frac{1}{n} \le \tau^{\boldsymbol{S}} := \tau(\boldsymbol{S}^X_{2n} + \boldsymbol{S}^Y_{2n}) \le \tau_n$ and that the $\boldsymbol{S}$-processes are increasing in time. It follows that

$$\begin{aligned} E'_{n,r} &= \left\{ \boldsymbol{S}^X_{2n}(s) \ge \boldsymbol{S}^Y_{2n}(s)\,\forall\, s < \tau_n,\ \boldsymbol{S}^X_{2n}(\tau_n) \ge \frac{1}{2},\ \boldsymbol{S}^Y_{2n}(\tau_n) \ge \frac{1}{2}, \tau_n = \frac{r}{2n} \right\} \\ &\supseteq \left\{ \boldsymbol{S}^X_{2n}(s) \ge \boldsymbol{S}^Y_{2n}(s)\,\forall\, s < \tau^{\boldsymbol{S}},\ \boldsymbol{S}^X_{2n}(\tau^{\boldsymbol{S}}) \ge \frac{1}{2},\ \boldsymbol{S}^Y_{2n}(\tau^{\boldsymbol{S}}) \ge \frac{1}{2}, \tau_n = \frac{r}{2n} \right\}. \end{aligned}$$

Denote by $\widetilde{E}_{n,r}$ the latter event and define $\widetilde{E}_n = \bigcup_{r \in \mathcal{I}^{(w)}_n} E_{n,r}$. We obtain

$$\mathbb{P}\left\{ (\sigma_n, \lambda_n, \tau_n) \in G;\ E_n \right\} \ge \frac{1}{4}\mathbb{P}\left\{ \frac{1}{\tau_n}\boldsymbol{L}^Y_{2n} \in B\left(\mu, \frac{\rho_1}{4}\right);\ \widetilde{E}_n \right\}.$$

Let now N_2 be large enough such that $n \ge N_2$ implies $n > \frac{2}{w} \vee \frac{2}{\rho_3} \vee \frac{4w^2}{\rho_1} \vee \frac{8}{w\rho_1}$ and $\frac{1}{m_1+2\rho_2} - \frac{1}{n} < \frac{1}{m_1+\rho_2}$. Then

$$\begin{aligned} \beta\left(\frac{1}{\tau_n}\boldsymbol{L}^Y_{2n}(\tau_n) - \frac{1}{\tau^{\boldsymbol{S}}}\boldsymbol{L}^Y_{2n}(\tau^{\boldsymbol{S}}) \right) &\le \beta\left(\frac{1}{\tau_n}\boldsymbol{L}^Y_{2n}(\tau_n) - \frac{1}{\tau^{\boldsymbol{S}}}\boldsymbol{L}^Y_{2n}(\tau_n) \right) \\ &\quad + \beta\left(\frac{1}{\tau^{\boldsymbol{S}}}\boldsymbol{L}^Y_{2n}(\tau_n) - \frac{1}{\tau^{\boldsymbol{S}}}\boldsymbol{L}^Y_{2n}(\tau^{\boldsymbol{S}}) \right) \\ &\le \left| \frac{1}{\tau_n} - \frac{1}{\tau^{\boldsymbol{S}}} \right| \beta\left(\boldsymbol{L}^Y_{2n}(\tau_n), 0 \right) + \frac{\left|\tau_n - \tau^{\boldsymbol{S}}\right|}{\tau^{\boldsymbol{S}}} \\ &< \frac{1}{nw} < \frac{\rho_1}{8} \end{aligned}$$

and further, using the fact that $\left|\tau_n - \tau^{\boldsymbol{S}}\right| < \frac{1}{n}$ repeatedly,

$$\left\{ \tau^{\boldsymbol{S}} \in \mathcal{I}^t \right\} \subseteq \left\{ \tau_n > \frac{w}{2}, \tau \in (t-\rho_3, t+\rho_3) \cap \left(\frac{1}{m+\rho_2}, \frac{1}{m-\rho_2} \right) \right.$$

where $\mathcal{I}^t = (w,\infty) \cap \left(t - \frac{\rho_3}{2}, t + \frac{\rho_3}{2}\right) \cap \left(\frac{1}{2m+4\rho_1}, \frac{1}{2m-2\rho_1} \right)$. It follows that

$$\begin{aligned} &\mathbb{P}\left\{ \frac{1}{\tau_n}\boldsymbol{L}^Y_{2n} \in B\left(\mu, \frac{\rho_1}{4}\right); \widetilde{E}_n \right\} \\ &\qquad \ge \mathbb{P}\left\{ \frac{1}{\tau^{\boldsymbol{S}}}\boldsymbol{L}^Y_{2n}\left(\tau^{\boldsymbol{S}}\right) \in B\left(\mu, \frac{\rho_1}{8}\right), \tau^{\boldsymbol{S}} \in \mathcal{I}^t, \left(\boldsymbol{S}^X_{2n}, \boldsymbol{S}^Y_{2n}\right) \in \mathcal{I}^{\boldsymbol{S}}_w \right\}. \end{aligned} \tag{3.4.15}$$

Here,

$$\mathcal{I}_w^{\boldsymbol{S}} = \{(x,y)\colon x(s) > y(s) - w \,\forall s < \tau(x+y),\, x(\tau(x+y)),\, y(\tau(x+y)) > \frac{1}{2} - w\}.$$

The right-hand side of (3.4.15) is of the form $(\boldsymbol{S}_{2n}^X, \boldsymbol{S}_{2n}^Y, \boldsymbol{S}_{2n}^X, \boldsymbol{S}_{2n}^X) \in U$ for an open subset U of $\mathcal{C}\left([0,1];\mathcal{Y}^2\right)$. So we can apply Corollary 3.2.11, then let $w \to 0$ and obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_n(G) \geq -2\inf\left\{ I(x,p,y,q)\colon \beta\left(\frac{1}{\tau}q(\tau),\mu\right) < \frac{\rho_1}{8}, \tau \in \mathcal{I}^\tau, (x,y) \in \mathcal{I}^{\boldsymbol{S}} \right\} \tag{3.4.16}$$

where $\tau := \tau(x+y)$ and $\mathcal{I}^{\boldsymbol{S}} = \{(x,y)\colon x(s) \geq y(s) \,\forall s < \tau,\, x(\tau) = y(\tau) = \frac{1}{2}\}$. Let $(x,p) \in \mathcal{E}$ be such that $x(s) = m_1(p(s)) \;\forall\, s$, for any $s \in [0,t)$, the inequality $x(s) \geq s m_1(\mu)$ holds and $x(t) = \frac{1}{2}$. Define further $\widetilde{q}\colon [0,1] \longrightarrow \mathfrak{M}_+(\mathbb{N})$ by

$$\widetilde{q}(s) = \begin{cases} s\mu & \text{if } s \in [0,t] \\[2ex] t\mu + (s-t)\mathfrak{G}_2 & \text{if } s \in [t,1] \end{cases}$$

and $\widetilde{y}(t) = m_1(\widetilde{p})$. Then (x,p), $(\widetilde{y},\widetilde{q}) \in \mathcal{E}$ and $\tau(x+\widetilde{y}) = t \in \mathcal{I}^t$. By construction $(x,\widetilde{y}) \in \mathcal{I}^{\boldsymbol{S}}$. Moreover, $\int_0^1 H\left(\dot{\widetilde{q}}(s)|\mathfrak{G}_2\right) \mathrm{d}s = tH(\mu|\mathfrak{G}_2)$ and (by convexity) $\int_0^1 H(\dot{p}(s)|\mathfrak{G}_2)\,\mathrm{d}s \geq tH(\frac{1}{t}p(t)|\mu)$. So by (3.4.16), and the same argument as for (3.4.10),

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log \nu_n(G) &\geq -2\inf\left\{ I\left(\boldsymbol{x},\boldsymbol{p},\widetilde{y},\widetilde{q}\right) : x(s) \leq s m_1(\mu),\, x(t) = \frac{1}{2} \right\} \\
&= -2\inf\left\{ t\left(H\left(\mu|\mathfrak{G}_2\right) + H\left(p|\mathfrak{G}_2\right)\right) : m_1(q) = \frac{1}{2} \right\} \\
&\geq -J(m,\mu,t).
\end{aligned}$$

This concludes the proof of the lower bound for $(\nu_n)_{n\in\mathbb{N}}$, and hence of Theorem 3.4.1.

□

3.5 A Formula for the Maximum of the Support

In this section we apply our large deviations result to a problem from free probability theory. Recall that the R-transform of a compactly supported probability measure μ is defined by

$$R_\mu(z) = K_\mu(z) - \frac{1}{z} = \sum_{n=0}^{\infty} k_{n+1}(\mu) z^n.$$

where K_μ is the local inverse of the *Cauchy transform* G_μ of μ around infinity, and that the coefficients $(k_n(\mu))_{n\in\mathbb{N}}$ are called the *free cumulants* of μ. If μ has compact support it is determined by its R-transform. So, given an R-transform we can, at least in theory, obtain the corresponding probability measure. However in order to do so one needs to find the functional inverse of $R(z) + \frac{1}{z}$ for which a closed-form expression may not exist. Using the large deviations principle of Section 3.4 we can deduce the right edge of the support of μ, provided that the free cumulants are non-negative.

As we saw in Section 2.2, the problem of determining a measure from its R-transform occurs in free probability: if a_1, a_2 are free non-commutative random variables of law μ_1, μ_2 respectively then the law μ of $a_1 + a_2$ has the property that $k_n(\mu) = k_n(\mu_1) + k_n(\mu_2)$ and the law ν of λa_1 has $k_n(\nu) = \lambda k_n(\mu_1)$ for any $\lambda \in \mathbb{R}$. This linearity property allows the computation of laws of free random variables, similarly to the moment generating function in commutative probability theory.

Because the R-transform determines the underlying probability measure one might still hope to recover some information about the measure, for example about the support, even when the Cauchy transform cannot be obtained explicitly. The special case where the underlying law is a free convolution of a semicircular law with another distribution has been studied deeply by P. Biane [16].

In this section we describe how the maximum of the support of μ can be deduced from the free cumulants.

Recall that the moments and free cumulants of μ are related by the *free moment-cumulant formula*:

$$m_n(\mu) = \int t^n \mu(dt) = \sum_{\pi \in \mathrm{NC}(n)} \prod_{j=1}^{\infty} k_j(\mu)^{B_j(\pi)} \tag{3.5.1}$$

where $B_j(\pi)$ is the number of blocks of size j in π.Our starting point is the observation that the edge of the support of a measure can be deduced from the logarithmic asymptotics of its moments: namely if ρ_μ is the maximum of the support of μ then

$$\log \rho_\mu = \limsup_{n\to\infty} \frac{1}{n} \log \int t^n \mu(\mathrm{d}t). \tag{3.5.2}$$

Suppose for the moment that all cumulants are positive (which is indeed the first case we will consider, in Section 3.5.1). Then we can re-write (3.5.1) as the expectation of an exponential functional of a uniformly random non-crossing partition. Namely, if $\theta\colon \mathbb{N} \longrightarrow \mathbb{R}$ is given by $\theta_j := \log k_j$, and $\widehat{\mathbb{E}}_n = \mathbb{E}(\cdot|E_n)$ (where E_n is the event we conditioned in Section 3.3) and C_n, the n^{th} Catalan number, is the cardinality of $\mathrm{NC}(n)$) then

$$\int t^n \mu(\mathrm{d}t) = \sum_{\pi \in \mathrm{NC}(n)} \exp\left(\sum_{j=1}^{\infty} \log(k_j) B_j(\pi)\right) = C_n\, \widehat{\mathbb{E}}_n\left[e^{2n\tau_n\,\langle \theta, \lambda_n\rangle}\right].$$

In Section 3.5.1 we evaluate the logarithmic asymptotics of this expectation by Varadhan's Lemma, using the large deviations principle we have proved above.

Using the fact that $\lim_{\epsilon\to 0} \epsilon \log \epsilon = 0$ one might suppose that a similar result will still hold when some of the cumulants are allowed to be zero. This is indeed the case and we will prove this in Section 3.5.2.

Remark 3.5.3. For $\gamma \in \mathbb{R}$ the shift operation given by $S_\gamma(\mu)(E) = \mu(\{x-\gamma \colon x \in E\})$ shifts the maximum of the support by γ to the right. Also $S_\gamma(\mu) = \mu \boxplus \delta_\gamma$ which leaves all cumulants invariant, except for the first which is incremented by γ. So we can always take the first cumulant to be anything we like.

3.5.1 All Free Cumulants Positive

We first consider the case where all free cumulants are positive. Examples include the free Poisson distribution.

Theorem 3.5.4. *Let μ be a compactly supported probability measure on $[0,\infty)$ such that its free cumulants $(k_j)_{j\in\mathbb{N}}$ all positive. Then the right edge ρ_μ of the support of μ is given by*

$$\log \rho_\mu = \sup\left\{ \frac{1}{m_1(p)} \sum_{m=1}^{\infty} p_m \log\left(\frac{k_m}{p_m}\right) + \frac{\Theta\big(m_1(p)\big)}{m_1(p)} \colon p \in \mathfrak{M}_1^1(\mathbb{N}) \right\} \tag{3.5.5}$$

where $\mathfrak{M}_1^1(\mathbb{N}) = \{p \in \mathfrak{M}_1(\mathbb{N}) \colon m_1(p) < \infty\}$ is the set of probability measures on $\mathbb{N}$ with finite mean and Θ was defined in (3.4.9).

This variational problem can often be solved by Lagrange multipliers or similar methods. Some examples are given below.

Remark 3.5.6. Equation (3.5.5) looks somewhat similar to *Varadhan's spectral radius formula* [37, Exercise 3.1.19], giving the spectral radius of a (deterministic) $N \times N$ matrix in terms of its entries. Namely, let $B = (b_{ij})_{i,j=1}^N$ be irreducible and have strictly positive entries then the spectral radius (absolutely largest eigenvalue) ρ_B of B is given by

$$\log \rho_B = \sup\left\{ \sum_{i,j=1}^{N} q(i,j) \log\left(\frac{b(i,j)}{q_f(j|i)}\right) \colon q \in \mathfrak{M}_1\left(\underline{N} \times \underline{N}\right), \sum_{j=1}^{N} q(\cdot,j) = \sum_{j=1}^{n} q(j,\cdot) \right\} \tag{3.5.7}$$

where $q_f(j|i) = \frac{q(i,j)}{\sum_r q(i,r)}$. Despite the apparent formal similarities we do not seem to be able to relate this formula to ours. This is because the free cumulants of a deterministic matrix are given by a very complicated function of its entries.

Proof of Theorem 3.5.4. By Stirling's formula, $\frac{1}{n}\log C_n \longrightarrow \log 4$ as $n \longrightarrow \infty$, so that

$$\limsup_{n\to\infty} \int t^n \,\mu(\mathrm{d}t) = \log 4 + \limsup_{n\to\infty} \frac{1}{n} \log \widehat{\mathbb{E}}_n \left[e^{ng(\lambda_n,\tau_n)} \right] \tag{3.5.8}$$

where $g\colon \mathfrak{M}_1(\mathbb{N}) \times [0,1] \longrightarrow \mathbb{R}$ is defined by $g(\mu,t) = 2t\langle\theta,\mu\rangle$.

It is a direct application of the contraction principle, Theorem 2.3.13, that $(\lambda_n,\tau_n)_{n\in\mathbb{N}}$ satisfies a large deviations principle on $\mathfrak{M}_1(\mathbb{N}) \times [0,1]$ with rate function $\widetilde{J}_{13}$ given by $\widetilde{J}_{13}(\mu,t) = J(m_1(\mu),\mu,t)$.

Suppose first that the sequence $(k_n)_{n\in\mathbb{N}}$ is bounded by $K \in (0,\infty)$. Then g is continuous and bounded, with norm $\|g\|_\infty \le 2\log K$. So for *any* $\gamma > 1$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \widehat{\mathbb{E}}_n \left[e^{n\gamma g(\tau_n,\lambda_n)} \right] \le 2\gamma \log(K) < \infty.$$

Hence the moment condition for Varadhan's Lemma (Theorem 2.3.11) applies and so

$$\lim_{n\to\infty} \frac{1}{n} \log \widehat{\mathbb{E}}_n \left[e^{ng(\lambda_n,\tau_n)} \right] = \sup \left\{ g(\mu,t) - \widetilde{J}_{13}(\mu,t) \colon (\mu,t) \in \mathfrak{M}_1(\mathbb{N}) \times [0,1] \right\}.$$

Let $\widehat{\rho}_\mu$ denote the left-hand side above and note that $\rho_\mu = \widehat{\rho}_\mu + \log 4$. So

$$\log(\rho_\mu) = \sup \left\{ \frac{1}{m} \sum_{n=1}^{\infty} p_n \log k_n + \frac{1}{m} H(p) - 2\log\left(1 - \frac{1}{m}\right) : m_1(p) = m \right\}$$

which is (3.5.5).

We now turn to the general case, that is, we remove the assumption that the sequence of free cumulants is bounded. Because μ is compactly supported, its R-transform is analytic on a neighbourhood of zero, by Theorem 3.2.1 in Hiai–Petz [55]. So there exist $\Gamma, R \in (0,\infty)$ with $k_n \le \Gamma R^n$ for all $n \in \mathbb{N}$. Define the dilation operator of scale $\frac{1}{R}$ by $D_{R^{-1}}(\mu)(A) = \mu\big(R^{-1}A\big)$, $\big($where $tA = \{tx \colon x \in A\}\big)$ and let $\widehat{k}_n = R^{-n}k_n$ be the n^{th} cumulant of $D_{R^{-1}}(\mu)$. The sequence $\left(\widehat{k}_n\right)_{n\in\mathbb{N}}$ is bounded, so the above applies to $\rho_{D_{R^{-1}}(\mu)} = \frac{\rho_\mu}{R}$. In particular,

$$\begin{aligned}\log \rho_{D_{R^{-1}}(\mu)} &= \sup\left\{\frac{1}{m}\sum_{n=1}^{\infty} p_n \log \widehat{k}_n + \frac{1}{m}H(p) + \frac{1}{m}\,\Theta(m) \colon m_1(p) = m\right\} \\ &= \sup\left\{\frac{1}{m}\sum_{n=1}^{\infty} p_n \log k_n + \frac{1}{m}H(p) + \frac{1}{m}\,\Theta(m) \colon m_1(p) = m\right\} - \log(R)\end{aligned}$$

which completes the proof of Theorem 3.5.4. $\square$

3.5.2 Non-Negative Free Cumulants

We now consider non-commutative random variables of which all free cumulants are non-negative but some of them are allowed to take the value zero. We will denote by L the set of $n \in \mathbb{N}$ such $k_n \neq 0$. As a prominent example we mention the centred semicircle distributions, where $L = \{2\}$.

It turns out that the variational formula (3.5.5) still holds, provided we follow the convention that $0 \log 0 = 0$.

Theorem 3.5.9. *Let μ be a compactly supported probability measure whose free cumulants $(k_n)_{n\in\mathbb{N}}$ are all non-negative. Then the maximum of the support ρ_μ of μ is given by*

$$\log(\rho_\mu) = \sup\left\{\frac{1}{m_1(p)}\sum_{n\in L} p_n \log\left(\frac{k_n}{p_n}\right) - \frac{\Theta(m_1(p))}{m_1(p)} \colon p \in \mathfrak{M}_1^1(L)\right\} \qquad (3.5.10)$$

where $\mathfrak{M}_1^1(L)$ denotes the set of $p \in \mathfrak{M}_1^1(\mathbb{N})$ such that $p(L^c) = 0$.

Proof. Since the set $\{p \in \mathfrak{M}_1(\mathbb{N}) \colon m_1(L^c) = 0\}$ is closed the direction '$\le$' in (3.5.10) follows directly from Exercise 2.1.24 in Deuschel–Stroock [39]. So we only need to show that the logarithm of the maximum of the support of our measure is bounded below by the variational formula. Let p be the free Poisson distribution with parameter 1 and recall that p has support $[0,4]$. For $\epsilon > 0$ let $\nu_\epsilon = D_{\epsilon^{-1}}(p)$, the ϵ-dilation of p (see the proof of Theorem 3.5.4). Then $k_n(\nu_\epsilon) = \epsilon^n$. By the remarks after Example 3.2.3 in [55] (page 98) the maximum of the support of $\mu_\epsilon := \mu \boxplus \nu_\epsilon$ is no bigger than the sum of those of μ and ν_ϵ. Moreover Theorem 3.5.4 applies to μ_ϵ so that

$$\begin{aligned}
\log(\rho_\mu) + 4\epsilon \geq \log(\rho_{\mu_\epsilon}) \\
&= \sup\left\{\frac{1}{m_1(p)}\sum_{n=1}^{\infty} p_n \log\left(\frac{k_n+\epsilon^n}{p_n}\right) + \frac{\Theta(m_1(p))}{m_1(p)} : p \in \mathfrak{M}_1^1(\mathbb{N})\right\} \\
&\geq \sup\left\{\frac{1}{m_1(p)}\sum_{n=1}^{\infty} p_n \log\left(\frac{k_n+\epsilon^n}{p_n}\right) + \frac{\Theta(m_1(p))}{m_1(p)} : p \in \mathfrak{M}_1^1(L)\right\} \\
&\geq \sup\left\{\frac{1}{m_1(p)}\sum_{n=1}^{\infty} p_n \log\left(\frac{k_n}{p_n}\right) + \frac{\Theta(m_1(p))}{m_1(p)} : p \in \mathfrak{M}_1^1(L)\right\}
\end{aligned}$$

using the fact that $\epsilon > 0$. Letting ϵ tend to zero yields the '$\geq$' direction of (3.5.10).

$\square$

3.6 Examples

We conclude this chapter with a few examples where our formula can be applied. The main requirement, that the free cumulants be non-negative, is satisfied in a wide range of cases.

Example 3.6.1. As a warm-up let us consider two (known) examples where the variational problem can be solved to give an explicit formula for the maximum of the support. The simplest example is the centred semicircle law of radius r. Then, in the notation of Section 3.5, $L = \{2\}$ and $k_2(\sigma_r) = \frac{r^2}{4}$. The only probability measure on L is δ_2 which has $m_1(\delta_2) = 2$. Therefore,

$$\begin{aligned}\log \rho_{\sigma_r} &= \frac{1}{2}\log k_2 + \frac{1}{2}\Theta(2)\\ &= \log\left(\frac{r}{2}\right) + \frac{1}{2}\left(2\log\left(1-\frac{1}{2}\right)\right) = \log(r).\end{aligned}$$

Next let $\lambda \geq 1$ and consider the *free Poisson distribution* p_λ with parameter λ, i.e.,

$$p_\lambda(\mathrm{d}t) = \frac{1}{2\pi t}\sqrt{4\lambda - (t-1-\lambda)^2}\,\mathbf{1}_{[(1-\sqrt{\lambda})^2,(1+\sqrt{\lambda})^2]}(t)\,\mathrm{d}t.$$

The free cumulants are given by $k_n = \lambda$ for all $n \in \mathbb{N}$ and therefore

$$\begin{aligned}\log \rho_{p_\lambda} &= \sup\left\{2\tau\log(\lambda) + 2\tau H(p) + 2\tau\Theta\left(\frac{1}{2\tau}\right) : m_1(p) = \frac{1}{2\tau}\right\}\\ &= 2\sup_{\tau\leq\frac{1}{2}}\left[\tau\log\lambda + 2\tau\Theta\left(\frac{1}{2\tau}\right)\right].\end{aligned}$$

Putting $\Psi_\lambda(\tau) = \tau\log\lambda + 2\tau\Theta\left(\frac{1}{2\tau}\right)$ we easily verify that $\Psi'_\lambda(\tau^*) = 0$ for $\tau^* = \frac{\sqrt{\lambda}}{2(\sqrt{\lambda}+1)}$ and that this critical point is the absolute maximum of Ψ_λ on $\left[0, \frac{1}{2}\right]$. Another direct computation yields $\log\rho_{p_\lambda} = 2\Phi_\lambda(\tau^*) = 2\log\left(1+\sqrt{\lambda}\right)$, i.e. $\rho_{p_\lambda} = \left(1+\sqrt{\lambda}\right)^2$.

Example 3.6.2. Let us consider $\mu = p \boxplus u$ where p is the free Poisson law of parameter 1 and u is the uniform distribution on $[-1, 1]$. This corresponds, for example, to the limiting distribution of $T_N^* T_N + \mathrm{diag}(\rho_1, \ldots, \rho_N)$ where T_N is an $N \times N$ real random matrix with i.i.d. entries of mean 0 and variance 1 and all moments bounded and $\rho_N(j) = j - 1 - \frac{N}{2}$. The R-transform of μ is given by

$$R_\mu(z) = R_p(z) + R_u(z) = \frac{1}{1-z} + \coth(z) - \frac{1}{z}$$

which cannot be inverted explicitly. We obtain an implicit equation for the maximum of the support of μ, i.e. the limiting largest eigenvalue:

$$\rho_\mu = \frac{\pi(m-1)}{m\gamma}$$

where (γ, m) is the unique pair of positive reals satisfying the equations

$$\frac{1}{m-1} = \frac{\gamma}{1-\gamma} + \coth(\gamma)$$
$$\frac{\lambda(m-1)}{1-\gamma} + (m-1)\coth(\gamma) = \frac{m-1}{\gamma} + \frac{\gamma^2 + (1-\gamma)^2}{m\gamma(1-\gamma)^2} + \frac{\gamma}{m}\left(1 - \coth^2(\gamma)\right).$$

3.6.1 Freely Infinitely Divisible Distributions

Let μ be *freely infinitely divisible.* That is, for every $n \in \mathbb{N}$ there exists a compactly supported probability law μ_n such that μ is the n-fold free convolution of μ_n with itself:

$$\mu = \underbrace{\mu_n \boxplus \ldots \boxplus \mu_n}_{n \text{ times}}.$$

Freely infinitely divisible probability measures have been studied by Barndorff-Nielsen – Thorbjørnsen [8, 9]. Many of their properties are non-commutative analogues of those enjoyed by classical infinitely divisible distributions, for example they lead to the concept of free Lévy processes. There exists an analogue of the Lévy-Khintchine representation, a version of which is given in [55], where Theorem 3.3.6 states that μ is freely infinitely divisible if and only if there exist $\alpha \in \mathbb{R}$ and a positive finite measure ν with compact support in $\mathbb{R}$ such that the R-transform R_μ of μ can be written, for z in a neighbourhood of $(\mathbb{C} \setminus \mathbb{R}) \cup \{0\}$, as

$$R_\mu(z) = \alpha + \int \frac{z}{1-xz}\,\nu(\mathrm{d}x). \qquad (3.6.3)$$

We call ν the *free Lévy–Khintchine measure* associated to μ. By Remark 3.5.3 we lose no generality by setting $k_1(\mu) = \alpha = 0$. Setting $m_0(\nu) := \nu(\mathbb{R})$ we can express the cumulants of μ in terms of the sequence $(m_n(\nu))_{n\geq 0}$ by observing that $k_n(\mu) = m_{n-2}(\nu)$ for $n \geq 2$.

So if μ is freely infinitely divisible and the moments of its free Lévy-Khintchine measure are all non-negative the variational formula for the maximum of the support of μ from Theorem 3.5.9 applies.

3.6.2 Series of Free Random variables

Let $\xi_1, \xi_2, \ldots$ be a sequence of free self-adjoint random variables of identical distribution μ_1 and consider the series

$$\xi = \sum_{n=1}^{\infty} n^{-\beta} \xi_n$$

where $\beta > 0$ is chosen large enough for the series to converge in the operator norm. Let $k_n(\mu_1)$ denote the free cumulants of μ_1 then the R-transform R_ξ of ξ is given by

$$R_\xi(z) = \sum_{n=1}^{\infty} n^{-\beta} R_{\xi_1}\left(n^{-\beta} z\right) = \sum_{n=1}^{\infty} n^{-\beta} \sum_{r=0}^{\infty} k_r(\mu_1) \left(n^{-\beta} z\right)^r .$$

Let U be a neighbourhood of zero where R_{ξ_1} is analytic then we have absolute convergence on U and hence may interchange the order of the two summations:

$$R_\xi(z) = \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} n^{-r\beta} k_r(\mu_1) z^r = \sum_{r=0}^{\infty} \zeta(\beta r) k_r(\mu_1) z^{r-1}$$

where ζ denotes the Riemann zeta function. So we conclude that the free cumulants of ξ are given in terms of those of ξ_1 by $k_n = \zeta(\beta n)\, k_n(\mu_1)$.

It may not be possible to locally invert the corresponding analytic function R_0 in closed form. In this case our formula comes in useful and we obtain:

Corollary 3.6.4. *Suppose that the free cumulants of μ_1 are all non-negative. Then the right edge ρ_0 of the support of the law of the series ξ_0 is given by*

$$\log(\rho_0) = \sup\left\{\frac{1}{m_1(p)}\sum_{n=1}^{\infty} p_n \log\left(\frac{\zeta(\beta n)k_n^{(0)}}{p_n}\right) - \frac{\Theta(m_1(p))}{m_1(p)} : p \in \mathfrak{M}_1^1(L)\right\}. \quad (3.6.5)$$

In some cases we can solve this variational problem and obtain a more or less explicit formula for the maximum of the support.

Example 3.6.6. Suppose μ_1 is the free Poisson distribution of parameter $\lambda \geq 1$. We set $\beta = 2$ and study $\sum_n n^{-2}\xi_n$ where the ξ_n are free and all distributed according to the free Poisson law. The corresponding R-transform is

$$R(z) = \frac{\lambda\left(1 - \sqrt{z}\cot\left(\sqrt{z}\right)\right)}{2z}$$

for which no closed-form inverse exists. However there is a unique maximiser for the corresponding variational problem (3.6.5), given by $p_n = \frac{\lambda\zeta(2n)}{Z}e^{mtn}$ and determined by its mean m. That mean is given implicitly by

$$\lambda(m-1) - 2 = \sqrt{4\lambda m^2 - 2\lambda m - 2(\lambda - 2)}\,\cot\left(\frac{\sqrt{4\lambda m^2 - 2\lambda m - 2(\lambda-2)}}{\lambda(m-1)}\right)$$

which has a unique solution m_* in the relevant interval. The right edge is therefore given by

$$\rho = \log\frac{\lambda^2 m_*^2(m_* - 1)}{4\lambda m_*^2 - 2\lambda m_* - 2(\lambda - 2)}.$$

The choice $\lambda = 1$ corresponds to the square integral of a free Brownian bridge which we will study in Chapter 4.

Another example, where the ξ_n are distributed according to the *commutator* of the standard semicircle law with itself, can also be found in Chapter 4. The commutator of two free random variables a and b is $[a, b] = i\,(ab - ba)$, see Nica–Speicher [82]. The free random variable $[a, b]$ is bounded and self-adjoint, provided a and b are. In Chapter 4 implicit equations for the maximum of the support of $[a, b]$ will be obtained when a and b are two free standard semicircular random variables.

Chapter 4

Functionals of the Free Brownian Bridge

4.1 Series Representations for the Bridge

We start off by pointing out free analogues of two representations of the classical Brownian bridge as series of independent Gaussian random variables. Series of free random variables were analysed in [10].

The first is the analogue of a representation by LÉVY [63]. Let $(e_n, f_n \colon n \in \mathbb{N})$ be an orthonormal sequence in the full Fock space $\mathcal{H}$ and define $\xi_n = s(e_n)$ and $\eta_n = s(f_n)$, so that $\{\xi_n, \eta_m \colon (n,m) \in \mathbb{N}^2\}$ is a set of free standard semicircular variables in $\mathcal{A}$.

Proposition 4.1.1. *The process $\beta_{2\pi}$ defined by*

$$\beta_{2\pi}(t) = \sum_{n=1}^{\infty} \frac{\cos(nt) - 1}{n\sqrt{\pi}} \xi_n + \sum_{n=1}^{\infty} \frac{\sin(nt)}{n\sqrt{\pi}} \eta_n \tag{4.1.2}$$

is a free Brownian bridge on $[0, 2\pi]$.

Proof. By continuity and linearity of the operator $s(\cdot)$ it follows that the right-hand side of (4.1.2) converges in $\mathcal{A}$ and that $\beta_{2\pi}(t)$ is a centred semircular variable. A direct computation verifies that $\beta_{2\pi}$ has the right covariance kernel. □

Next we show how KAC's representation [56] for the classical Brownian bridge on the unit interval can be translated into the setting of free probability. His method extends to all centred semicircular (or indeed Gaussian) processes, as follows. Everything relies on the following classical result from functional analysis, see BOLLOBAS [25]

Theorem 4.1.3 (Mercer's theorem)**.** *Let $K\colon [0,1] \times [a,b] \longrightarrow \mathbb{R}$ be a non-negative definite symmetric kernel. Let T_K be the operator on $\mathcal{H}$ associated to K, that is,*

$$T_K(f)(s) = \int_0^1 K(s,t)\, f(t)\, \mathrm{d}t. \tag{4.1.4}$$

Then there exists an orthonormal basis $(f_n)_{n\in\mathbb{N}}$ of $\mathrm{L}^2[0,1]$ consisting of eigenfunctions of T_K such that the corresponding eigenvalues λ_n are non-negative, $f_n \in \mathcal{C}[0,1]$ whenever $\lambda_n \neq 0$ and

$$K(s,t) = \sum_{n=1}^{\infty} \lambda_n f_n(s) f_n(t) \tag{4.1.5}$$

where the convergence is absolute and uniform, and hence also in $\mathrm{L}^2[0,1]$.

We can use Mercer's theorem to represent any centred semicircular process as a series of free standard semicircular random variables, noting that if Y is a centred semicircular process indexed by $[0,1]$ then its covariance function K defined by $K(s,t) = \phi(Y(s)Y(t))$ is a non-negative kernel on $[0,1]$ which is also symmetric, by traciality of ϕ.

Corollary 4.1.6. *Let $K, \mathcal{H}, (\lambda_n, f_n)_{n\in\mathbb{N}}$ be as in Mercer's theorem and let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of free standard semicirculars, defined in terms of creation and anni-*

hilation operators as in Section 2.2.8. Then the process Y defined by

$$Y(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} f_n(t)\, \eta_n \tag{4.1.7}$$

is a centred semicircular process of covariance K.

Proof. As before convergence in the operator topology of $\mathcal{A}$ follows from linearity and continuity of the operator $s(\cdot)$. Further it is once more immediate that Y is a centred semircircular process. Its covariance kernel is given by

$$\begin{aligned}\phi(Y(s)Y(t)) &= \sum_{m,n=1}^{\infty} \sqrt{\lambda_m \lambda_n} f_m(s) f_n(t) \phi(\eta_m \eta_n) \\ &= \sum_{n=1}^{\infty} \lambda_n f_n(s) f_n(t) = K(s,t)\end{aligned}$$

by Mercer's theorem. □

For the free Brownian bridge on $[0,1]$ we have $K(s,t) = s \wedge t - st$. Solving the corresponding eigenvalue-eigenvector equation we obtain Kac's representation in the free setting.

$$\beta_1(t) = \sum_{n=1}^{\infty} \frac{\sqrt{2}\sin(n\pi t)}{n\pi}\, \eta_n. \tag{4.1.8}$$

4.2 Square Norm of the Free Brownian Bridge

In this section we consider the square-norm of a free Brownian bridge β on interval. Recall that $\mathcal{A}$ is a C*-algebra so that we can consider β as a map from $[0,1]$ into a Banach space which is easily seen to be continuous. We can therefore use Riemann integration to define

$$\Gamma = \int_0^1 \beta(t)^2 \, \mathrm{d}t$$

where β is a free Brownian bridge on $[0,1]$. In this section we discuss the distribution of the non-commutative random variable Γ, using the representation (4.1.8). KAC [56] showed that the Laplace transform of the commutative analogue of Γ is given by

$$\widehat{f}(p) = \left(\frac{\sqrt{2p}}{\sinh \sqrt{2p}} \right)^{(1/2)} .$$

Other properties, in particular the density function f, were computed, most recently by TOLMATZ [117].

We give here the R-transform of Γ and an expression for its moments involving a sum over non-crossing partitions. Further below we show that the distribution μ_Γ of Γ is freely infinitely divisible. This gives us some analytic tools to show that there exist $a, b \in [0, \infty)$ with $a < b$ such that the support of μ_Γ is $[a, b]$ and that μ_Γ has a smooth positive density on $[a, b]$. We give an implicit equation and a sketch for the density.

Finally we use a result from Chapter 3 to characterise the maximum b of the support of μ_Γ. In particular we show that $b < \frac{1}{2}$.

4.2.1 The R-transform

The Kac representation of semicircular process is well suited for computing quadratic functionals. Let Y be a semicircular process with covariance kernel K and series representation as in Corollary 4.1.6. By orthonormality of the eigenfunctions,

$$\int_0^1 Y^2(s) \, \mathrm{d}s = \sum_{n=1}^{\infty} \lambda_n \eta_n^2 .$$

Now the distribution of η_n^2 is well-known: the square of a standard semicircular random variable is a free Poisson element of unit rate and jump size (Nica–Speicher [83], Proposition 12.13). So the free cumulants of η_n^2 are all equal to 1 and hence its R-transform is given by $R_n(z) = \frac{1}{1-z}$, see [83, p. 205].

Using the properties of the R-transform mentioned in Remark 2.2.16 we can now compute the R-transform of $\int_0^1 Y^2(s)\,\mathrm{d}s$. In the case where Y is a free Brownian bridge we obtain the following

Proposition 4.2.1. *The R-transform of the square norm Γ of the free Brownian bridge is given by*

$$R_\Gamma(z) = \frac{1-\sqrt{z}\cot(\sqrt{z})}{2z}. \tag{4.2.2}$$

Proof. The eigenvalues of K are given by $\lambda_n = \frac{1}{n\pi}$. So for $|z| < \pi^2$ we have

$$R_\Gamma(z) = \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} R_n\left(\frac{z}{\pi^2 n^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 - z} = \frac{1-\sqrt{z}\cot(\sqrt{z})}{2z}$$

as claimed. □

The free cumulants of Γ are therefore given by

$$k_m = \frac{\zeta(2m)}{\pi^{2m}} = (-4)^{m+1}\,\frac{B_{2m}}{2(2m)!}$$

where B_n is the n^{th} Bernoulli number and ζ the Riemann zeta function. With (2.2.9) we obtain a formula for the moments involving a sum over non-crossing partitions:

$$\phi\left(\Gamma^n\right) = \frac{1}{\pi^{2n}} \sum_{\sigma\in\mathrm{NC}(n)} \prod_{r=1}^{m_\sigma} \zeta(2l_r^\sigma) = (-4)^n \sum_{\sigma\in\mathrm{NC}(n)} \prod_{r=1}^{m_\sigma} \frac{B_{2l_r^\sigma}}{(2l_r^\sigma)!}$$

where m_σ denotes the number of equivalence classes of a non-crossing partition σ

and l_r^σ is the size of the r^{th} equivalence class of σ.

While there does not seem to exist a closed-form expression for the inverse of $K_\Gamma(z) = R_\Gamma(z) - \frac{1}{z}$ (and hence, by the Stieltjes inversion formula, for the density) we will describe some properties of the law μ_Γ of Γ. We will prove that μ_Γ is freely infinitely divisible, has a positive analytic density on a single interval and give an equation for the right end point of that interval.

4.2.2 Free Infinite Divisibility

The concept of infinite divisibility has a natural analogue in free probability theory. Noting that the square norm of the free Brownian bridge is freely infinitely divisible we will use the approach of P. Biane in his appendix to the paper [12] to prove that the law of Γ has a smooth density on its support and give an implicit formula for that density.

Definition 4.2.3. A compactly supported probability measure μ is said to be *freely infinitely divisible* (or $\boxplus$-*infinitely divisible*) if for every $n \in \mathbb{N}$ there exists a compactly probability measure μ_n such that

$$\mu = \mu_n^{\boxplus n} = \underbrace{\mu_n \boxplus \ldots \boxplus \mu_n}_{n \text{ times}}$$

where $\boxplus$ denotes free convolution (Section 2.2).

Since for each n the free random variable η_n^2 has a free Poisson distribution and is therefore freely infinitely divisible it follows that Γ is also $\boxplus$-infinitely divisible.

Recall that the Cauchy transform G_Γ of Γ is an analytic map from the upper half plane $\mathbb{C}^+$ into the lower half plane $\mathbb{C}^-$, which is locally invertible on a neighbourhood of infinity, and that its local inverse is given by the K-transform K_Γ where

$$K_\Gamma(z) = R_\Gamma(z) + \frac{1}{z} = \frac{3 - \sqrt{z}\cot\left(\sqrt{z}\right)}{2z}.$$

From Proposition 5.12 in Bercovici–Voiculescu [13] and the infinite divisibility of Γ it is straightforward to deduce the following result.

Lemma 4.2.4. *The law μ_Γ of the square norm of the free Brownian bridge can have at most one atom. Moreover its Cauchy transform G_Γ is an analytic injection from $\mathbb{C}^+$ whose image is the connected component Ω in $\mathbb{C}^-$ of*

$$\widehat{\Omega} = \{z \in \mathbb{C}^- : \mathfrak{Im}\,(K_\Gamma(z)) > 0\}$$

that contains iy for small values of y.

It will be useful to characterise the boundary $\partial\Omega$.

Lemma 4.2.5. *For every $t \in (\pi, 2\pi)$ there exists unique $r(t) > 0$ such that the imaginary part of $(K_\Gamma(r(t)e^{it}))$ vanishes. Moreover*

$$\frac{\partial}{\partial z}\mathfrak{Im}K_\Gamma(z)\bigg|_{z=r(t)e^{it}} \neq 0 \qquad \forall\, t \in (\pi, 2\pi). \tag{4.2.6}$$

Proof. Fix $t \in (\pi, 2\pi)$. The imaginary part of K_Γ can be written in polar coordinates by

$$h_t(r) := \mathfrak{Im}K_\Gamma\left(r\, e^{it}\right) = -\frac{3\sin(t)}{2r} + \frac{\gamma\sinh(\sigma\sqrt{r})\cosh(\sigma\sqrt{r}) + \sigma\sin(\gamma\sqrt{r})\cos(\gamma\sqrt{r})}{2\sqrt{r}\left(\sin^2(\gamma\sqrt{r}) + \sinh^2(\sigma\sqrt{r})\right)}$$

where $\sigma = \sin(t/2)$ and $\gamma = \cos(t/2)$. Define $g_t(r) = 2r\, h_t(r^2)$. Then

$$g_t(r) = -\frac{6\sigma\gamma}{r} + \frac{\sigma\sin(2\gamma r) + \gamma\sinh(2\sigma r)}{2\left[\sin^2(\gamma\sqrt{r}) + \sinh^2(\sigma\sqrt{r})\right]}.$$

The function g_t blows up to $+\infty$ as $r \downarrow 0$. In particular g_t is strictly positive on $(0, R_2(t))$ for some $R_2(t) > 0$. Further there must be $R_1(t) > 0$ with $g_t'(r)$ negative on $(0, R_1(t))$. Splitting into the three cases whether $t \in \left(\pi, \frac{3\pi}{2}\right)$ or $t \in \left[\frac{3\pi}{2}, \frac{5\pi}{3}\right]$ or $t \in \left(\frac{5\pi}{3}, 2\pi\right)$ we can check directly that there exists $R_3(t) \in (0, R_1(t))$ such that $g_t(r) < 0$ for all $r > R_3(t)$. Further details on this lengthy but elementary computation can be found in Appendix A.

Hence g_t has a unique zero ρ_t, which must lie in $(R_2(t), R_3(t)) \subset (0, R_1(t))$. Hence $g_t'(\rho(t)) < 0$ and the result follows. $\square$

Therefore $\widehat{\Omega}$ is actually simply connected: it is given by the area enclosed by the real axis and the curve $\lambda = \{r_t e^{it} \colon t \in (\pi, 2\pi)\}$. In particular $\Omega = \widehat{\Omega}$ and $\partial\Omega$ is a continuous simple curve. So Carathéodory's theorem applies, wherefore the analytic bijection $G_\Gamma \colon \mathbb{C}^+ \longrightarrow \Omega$ extends to a homeomorphism (denoted $\widehat{G}_\Gamma$) from $\mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$ to the closure $\overline{\Omega}$ of Ω in $\mathbb{C} \cup \{\infty\}$.

Since Ω is bounded, so is its closure, whence $\widehat{G}_\Gamma$ is finite on $\mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$. The set of isolated points of the support of μ_Γ is exactly the set of $t \in \mathbb{R}$ such that $\widehat{G}_\Gamma(t) = \infty$ so $\mathrm{spt}(\mu_\Gamma)$ must be an interval $[a, b]$. From the Stieltjes inversion formula (see for example [55], p.93) it now follows that if we put for $x \in [a, b]$

$$\Phi(x) = -\frac{1}{\pi} \lim_{y \to 0} \mathfrak{Im}\left(G_\Gamma(x + iy)\right) = -\frac{1}{\pi} \mathfrak{Im}\left(\widehat{G}_\Gamma(x)\right) \tag{4.2.7}$$

then μ_Γ has density Φ with respect to Lebesgue measure on $[a, b]$. Since K_Γ is the inverse of G_Γ and because of (4.2.6) the implicit function theorem applies and hence Φ is smooth on $[a, b]$. Moreover it follows that

$$\mathrm{spt}\,\mu_\Gamma = K_\Gamma\left(\partial\Omega \cap \mathbb{C}^-\right) = \left[K_\Gamma\left(r_{\pi+}\right) \wedge K_\Gamma\left(r_{2\pi-}\right), K_\Gamma\left(r_{\pi+}\right) \vee K_\Gamma\left(r_{2\pi-}\right)\right].$$

where $r_{\pi+} = \lim_{s \downarrow 0} r_{\pi+s}$ and $r_{2\pi-} = \lim_{s \downarrow 0} r_{2\pi-s}$.

The operator Γ is positive so the support of μ_Γ must be contained in $[0, \infty)$. (We will show below that in fact the support is contained in $[0, 1/2]$.) Let us summarise the results of this section.

Proposition 4.2.8. *There exist $b > a \geq 0$ and a positive smooth function $\Phi \colon [a, b] \longrightarrow$*

$\mathbb{R}$ *such that*

$$\mu_\Gamma(\mathrm{d}t) = \Phi(t)\mathbf{1}_{[a,b]}(t). \tag{4.2.9}$$

The function Φ *is given by* $\Phi(x) = -\frac{1}{\pi}r(\tau_x)\sin(\tau_x)$ *where* $\tau_x \in (\pi, 2\pi)$ *is the unique solution to* $K_\Gamma\left(r(\tau_x)\, e^{i\tau_x}\right) = x$.

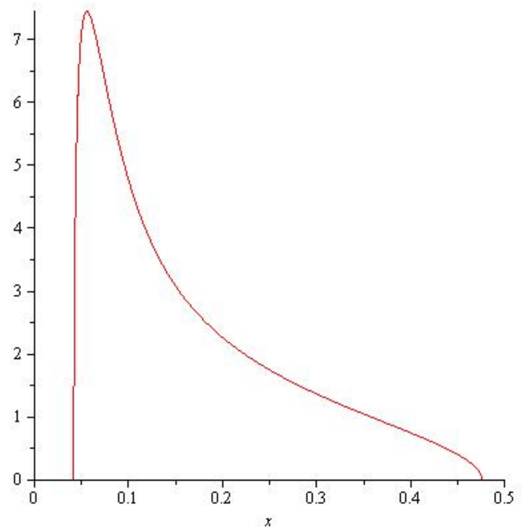

Figure 4.1: Sketch of the density of the L^2-norm of the free Brownian bridge, based on numerical computations

4.2.3 The Maximum of the Support

We now study the maximum of the support of μ_Γ, for which we will apply Theorem 3.5.4. It turns out the variational problem given by the theorem can be solved using the method of Lagrange multipliers. There exists a unique maximiser p^* for the supremum on the right-hand side of (3.5.5). Using the series expansion of $\zeta(2n)$ and interchanging summation we obtain

$$p_n^* = \frac{1}{m^* - 1}\,\zeta(2n)\left(\frac{\gamma}{\pi}\right)^{2n}$$

where γ is a rational function of m^* and m^* is the unique solution on $\left(\frac{3}{2}, \infty\right)$ of the equation

$$m - 3 = \sqrt{4m^2 - 2m - 6} \cot\left(\frac{\sqrt{4m^2 - 2m - 6}}{m - 1}\right) \tag{4.2.10}$$

After some computations which are detailed in Appendix A, we obtain an implicit equation for the right edge of the support of μ_Γ:

Proposition 4.2.11. *The number b from Proposition 4.2.8 is given by*

$$b = \frac{(m^*)^2 - m^*}{4(m^*)^2 - 2m^* - 6}$$

where m^* is the unique solution of (4.2.10) *on* $\left(\frac{3}{2}, \infty\right)$.

Remark 4.2.12. The function $B\colon m \longmapsto \frac{m^2 - m}{4m^2 - 2m - 6}$ is strictly decreasing on $\left(\frac{3}{2}, 2\right)$. Since the left-hand side of (4.2.10) is bigger than the right-hand side for $m = \frac{8}{5}$ but smaller for $m = 2$ it follows that $m^* \in \left(\frac{8}{5}, 2\right)$ and hence $b \leq B\left(\frac{8}{5}\right) < \frac{1}{2}$. It follows that the support of μ_Γ is contained in $\left[0, \frac{1}{2}\right]$.

Remark 4.2.13. Using the implicit characterisation of the density as in section 4.2.2 one could obtain, via a lengthy computation, another implicit characterisation for b.

4.3 The Signature of the Free Brownian Bridge

4.3.1 Signature and Rough Paths

In T. Lyons's paper [66] a new approach to differential equations driven by rough paths is proposed. For a general Banach-valued path $p\colon \mathbb{R}_+ \longrightarrow E$ we define, when this makes sense, the *signature* of p to be the process $S(p)$ taking values in the tensor algebra $T((E)) = \bigoplus_{n=0}^{\infty} E^{\otimes n}$ whose n^{th} component is given by the n-times iterated integral against p:

$$S(p)_n(t) = \int_{0<t_1<\ldots<t_n<t} \mathrm{d}p\,(t_1) \otimes \ldots \otimes \mathrm{d}p\,(t_n).$$

The signature is then used to solve general differential equations of the form

$$\mathrm{d}\,S(t) = S(t) \otimes \mathrm{d}p\,(t).$$

In order to show that this works if the path in question is a free Brownian motion X, Capitaine–Donati-Martin [31] define an integral of a class of suitable processes $\mathfrak{P}$ against X that yields a process taking values in the tensor product $\mathcal{A} \otimes \mathcal{A}$ and prove that X itself is contained in $\mathfrak{P}$. The integral is defined taking Riemann-type approximations, so it is straightforward to extend it to processes with finite variation. Using Remark 2.2.36 we can therefore define the *second component of the signature* of a free Brownian bridge β on $[0, 2\pi]$ by

$$Z(t) = \int_0^t \beta \otimes \mathrm{d}\,\beta \qquad t \in [0, 2\pi]$$

where the integral is in the sense of [31], see also Victoir [122].

If $\mathcal{A}$ is a von Neumann algebra and ϕ a faithful tracial state on $\mathcal{A}$ then its tensor product $\phi \otimes \phi$ is a faithful tracial state on the von Neumann tensor product $\mathcal{A} \otimes \mathcal{A}$ of $\mathcal{A}$ with itself, see for example [122], p. 109. So we can consider $(\mathcal{A} \otimes \mathcal{A}, \phi \otimes \phi)$ as a non-commutative probability space in its own right. We will discuss here the law of $Z(2\pi)$ with respect to this space.

We will also use the notation $\widehat{\mathcal{A}}$, $\widehat{\phi}$ for $\mathcal{A} \otimes \mathcal{A}$, $\phi \otimes \phi$ respectively.

4.3.2 Using the Lévy Representation

The representation (4.1.2) and a straightforward calculation using orthogonality of the trigonometric functions yield

Proposition 4.3.1. *The Lévy area of the free Brownian bridge at time 2π has the same law as the random variable*

$$Z(2\pi) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\xi_n \otimes \eta_n - \eta_n \otimes \xi_n\right). \tag{4.3.2}$$

Since ξ_n, η_n have symmetric distributions, so do $\xi_n \otimes \eta_n$ and $\eta_n \otimes \xi_n$. Hence the R-transform of $Z(2\pi)$ is given by

$$R_{Z(2\pi)}(z) = 2 \sum_{n=1}^{\infty} \frac{1}{n} R_{\xi\otimes\eta}\left(\frac{z}{n}\right). \tag{4.3.3}$$

Remark 4.3.4. By the definition of $\widehat{\phi}$ we have $\widehat{\phi}\left((\xi \otimes \eta)^k\right) = \phi(\xi^k)^2$ for $k \in \mathbb{N}$. Recall that $R_a(z) = \sum_{m=0}^{\infty} k_{m+1}(a) z^m$ where $k_m(a)$ denotes the m^{th} cumulant of a. In particular $k_1(\xi \otimes \eta) = \phi(\xi)^2 = 0$ so that (on a neighbourhood of zero) $R_{\xi\otimes\eta}(z) = zP(z)$ for some analytic function P. Rewriting (4.3.3) yields

$$R_{Z(2\pi)}(z) = 2z \sum_{n=1}^{\infty} \frac{1}{n^2} P\left(\frac{z}{n}\right), \tag{4.3.5}$$

in particular the right hand side of (4.3.3) converges in a neighbourhood of zero.

4.3.3 The Distribution of the Tensor Product and Meanders

We proceed to compute the R-transform of $\zeta := \xi \otimes \eta$ with ξ, η free standard semicirculars. Recall that the odd moments of ξ vanish and that $\phi(\xi^{2n})$ is given by the n^{th} *Catalan number*

$$\phi(\xi^{2n}) = C_n := \frac{1}{2n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}. \tag{4.3.6}$$

Since ξ, η are self-adjoint, so is ζ. Hence its law is a probability measure ν with compact support in $\mathbb{R}$. In particular ν is determined by its moments which are given by

$$\int t^m \nu(\mathrm{d}t) = \phi\left((\xi \otimes \eta)^m\right) = \phi(\xi^m)\phi(\eta^m) = \begin{cases} (C_k)^2 & \text{if } m = 2k \\ 0 & \text{if } m \text{ is odd} \end{cases} \tag{4.3.7}$$

i.e. ν is the law of $\zeta_1\zeta_2$ where the ζ_i are independent commutative random variables with standard semicircular distribution. Therefore ν is absolutely continuous with respect to Lebesgue measure with density ϕ given by

$$\phi(u) = \frac{1}{4\pi^2}\int_{-2}^{2} \sqrt{4-s^2}\,\sqrt{4-\left(\frac{u}{s}\right)^2}\;\mathbf{1}_{[-2,2]}\left(\frac{u}{s}\right)\frac{\mathrm{d}s}{s}. \tag{4.3.8}$$

The Catalan numbers C_n are well-known in combinatorics. They give, for example, the number of Dyck paths of length $2n$. Similarly there is a combinatorial interpretation of the squares of the Catalan numbers, as detailed in Lando–Zvonkin LandoZvonkin93 and Di Francesco–Golinelli–Guitter [46]: consider an infinite line in the plane and call it the *river*. A *meander* of order n is a closed self-avoiding connected loop intersecting the line through $2n$ points (the *bridges*). Two meanders are said to be *equivalent* if they can be deformed into each other by a smooth transformation without changing the order of the bridges. If a meander of order n consists of k closed connected non-intersecting (but possibly interlocking) loops it is said to have k *components*.

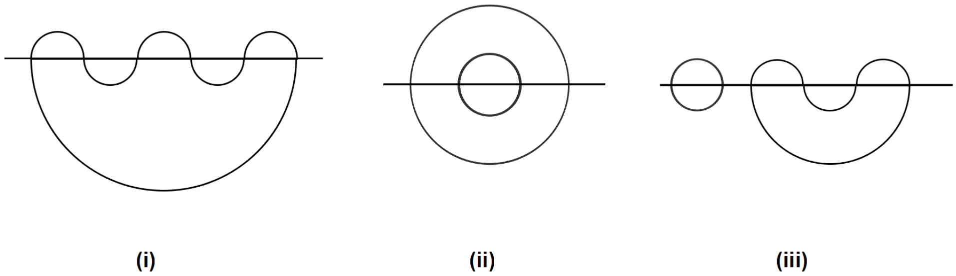

Figure 4.2: **(i)** 1-component meander of order 3; **(ii)** order 2, 2 components; **(ii)** order 3, 2 components

A multi-component meander is said to be *k-reducible* if a proper non-trivial collection of its connected components can be detached from the meander by cutting the river k times between the bridges. Otherwise the meander is said to be *k-irreducible*.

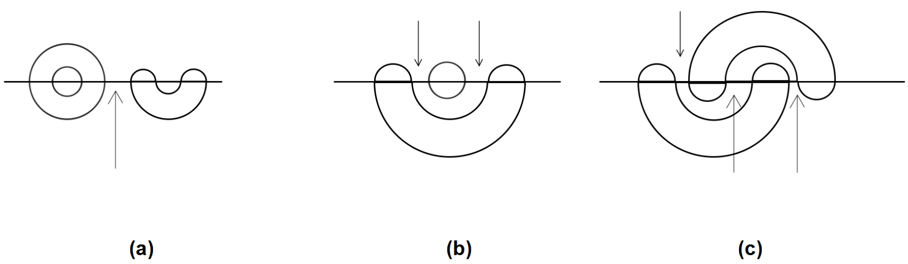

Figure 4.3: meanders that are **(a)** 1-reducible but 2-irreducible; **(b)** 1- and 2-reducible but 3-irreducible **(c)** 3-reducible

The 2-irreducible meanders have been studied extensively in [62] (where they are called irreducible meanders). Our connection to these objects is the following

Proposition 4.3.9. *Let q_n denote the number of 2-irreducible meanders of order $2n$ and $k_n = k_n(\xi \otimes \eta)$ the n^{th} cumulant of $\xi \otimes \eta$. Then*

$$k_n(\xi \otimes \eta) = \begin{cases} q_m & \text{if } n = 2m \\ 0 & \text{if } n \text{ is odd} \end{cases} \tag{4.3.10}$$

Proof. We first prove by induction that $k_n = 0$ if n is odd, which will follow from the fact that $\widehat{\phi}((\xi \otimes \eta)^n) = 0$ for n odd. Assume that $k_m = 0$ whenever $m < n$ is odd. From (2.2.9) it follows that

$$k_n = - \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi \neq \mathbf{1}}} k_\pi$$

where $k_\pi = k_{V_1} \dots k_{V_r}$ if $V_1, \dots, V_r$ are the equivalence classes of π and $\mathbf{1}$ denotes the identity partition, i.e. $[k]_{\mathbf{1}} = \underline{n}$. Every $\pi \in \mathrm{NC}(n) \setminus \{\mathbf{1}\}$ must contain at least one equivalence class of size m for some odd integer $m < n$. Since k_m is a factor of k_π and $k_m = 0$, the inductive hypothesis implies $k_n = 0$ as required. Hence

$$R_{\xi \otimes \eta}(z) = \sum_{n=1}^{\infty} k_{2n} z^{2n-1}.$$

Define the *moment series* of $\xi \otimes \eta$ by

$$M(z) = \frac{1}{z} G\left(\frac{1}{z}\right) = 1 + \sum_{n=1}^{\infty} \widehat{\phi}\left((\xi \otimes \eta)^n\right) z^n.$$

It is a consequence of the relationship between Cauchy and R-transform that

$$M(z) = 1 + zM(z)\, R(zM(z)). \tag{4.3.11}$$

We will introduce one more generating series. Put

$$\rho(z) = \sum_{n=1}^{\infty} q_n z^{2n-1}.$$

From (7.10) in [46] we have

$$M(z) = 1 + zM(z)\,\rho(zM(z)). \tag{4.3.12}$$

Combining (4.3.11) and (4.3.12) yields $\rho = R$ as power series. That $k_{2n} = q_n$ now follows from comparing coefficients. □

4.3.4 The Distribution of the Signature

So we have an explicit expression for the R-transform of $\xi \otimes \eta$. We will use this to obtain the R-transform of $Z(2\pi)$.

Recall that all odd cumulants of $\xi_n \otimes \eta_n$ and $\eta_n \otimes \xi_n$ vanish, hence the same is true of $Z(2\pi)$.

Proposition 4.3.13. *The $2n^{th}$ cumulant of $Z(2\pi)$ is $2\,\zeta(2n)\,q_n$ where ζ is the Riemann zeta function.*

Proof. Recall that $\zeta(m) = \sum_{n=1}^{\infty} n^{-m}$. So

$$\begin{aligned} R_{Z(2\pi)}(z) &= 2\sum_{n=1}^{\infty} \frac{1}{n}\, R_{\xi\otimes\eta}\left(\frac{z}{n}\right) = 2\sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} k_m \left(\frac{z}{n}\right)^{m-1} \\ &= 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^{-2m} q_m z^{2m-1} \\ &= \sum_{m=1}^{\infty} 2\,\zeta(2m) q_m z^{2m-1} \end{aligned}$$

where interchanging the sums over m and n is justified by absolute convergence. □

Definition 4.3.14 (see [112], p. 107)**.** Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be two sequences with generating functions f, g respectively. The *Hadamard product* of f, g is defined to be the generating function of $(a_n b_n)$, denoted $f \circledast g$. That is

$$f \circledast g(z) = \sum_{n=1}^{\infty} a_n b_n z^n.$$

So $R_{Z(2\pi)}$ is twice the Hadamard product of the generating functions of the 2-irreducible meanders and that of the sequence $\{\zeta(2m) \colon m \in \mathbb{N}\}$.

From (6.3.14) in Abramowitz–Stegun [1] we have for $|z| < 1$,

$$\sum_{n=2}^{\infty} \zeta(n+1) z^n = -\gamma - \Psi(1-z)$$

where γ is the Euler constant and Ψ is the *Digamma function* defined by

$$\Psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Since the generating series can be considered as functions inside their radius of convergence, we can use complex analysis to compute their Hadamard product. Namely

Lemma 4.3.15. *Let f, g be generating functions of $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and suppose that they are analytic on a neighbourhood of 0. Then*

$$(f \circledast g)(z^2) = \frac{1}{2\pi i} \int_\gamma f(zw)\, g\left(\frac{z}{w}\right) \frac{\mathrm{d}w}{w} \tag{4.3.16}$$

on a neighbourhood U of 0, where γ is a smooth closed curve around 0 and contained in U.

Proof. Let U_1, U_2 be neighbourhoods of 0 on which f and g respectively are analytic.

Then for $z \in U = U_1 \cap U_2$,

$$\begin{aligned} \frac{1}{2\pi i}\int_\gamma f(zw)\, g\left(\frac{z}{w}\right)\, \frac{\mathrm{d}w}{w} &= \left[f(z\eta)\, g\left(\frac{z}{\eta}\right)\right]_{\eta^0} \\ &= \left[\sum_{n=0}^{\infty} a_n (z\eta)^n \sum_{m=0}^{\infty} b_m \left(\frac{z}{w}\right)^m \right]_{\eta^0} \\ &= \left[\sum_{m,n} a_n b_m z^{n+m} \eta^{n-m} \right]_{\eta^0} \\ &= \sum_{n=0}^{\infty} a_n b_n z^{2n} = f \circledast g(z^2) \end{aligned}$$

where $[\cdot]_{\eta^0}$ denotes the constant term in a Laurent series in η. □

Corollary 4.3.17. *Let $\epsilon \in (0, \rho)$ where ρ is the radius of convergence of $R_{Z(2\pi)}$ and choose the canonical branch of the square root on $B(0, \rho)$. Then for $z \in B(0, \rho)$*

$$R_{Z(2\pi)}(z) = -\frac{z^{1/2}}{\pi i} \int_\Gamma \Psi(1 - z^{1/2} w)\, \mathcal{Q}\left(\frac{z}{w}\right)\, \mathrm{d}w \tag{4.3.18}$$

where $\Gamma = \partial B(0, \epsilon)$ and $\mathcal{Q}$ is the generating series of the q_m (recall that q_m denotes the number of 2-irreducible meanders of order $2n$).

Proof. By Proposition 4.3.13 we have $R_{Z(2\pi)} = 2\, \mathcal{Q} \circledast \Lambda$ where, using (4.3.16)

$$\Lambda(z) = \sum_{n=1}^{\infty} \zeta(m) z^m = -z\Psi(1-z) - \gamma z.$$

Lemma 4.3.15 now yields

$$\begin{aligned} (\mathcal{Q} \circledast \Lambda)(z^2) &= \frac{1}{2\pi i} \int_\Gamma \Lambda(zw)\, \mathcal{Q}\left(\frac{z}{w}\right)\, \frac{\mathrm{d}w}{w} \\ &= -\frac{1}{2\pi i} \int_\Gamma zw\, (\Psi(1 - zw) + \gamma)\ \mathcal{Q}\left(\frac{z}{w}\right)\, \frac{\mathrm{d}w}{w} \\ &= -\frac{1}{2\pi i} \int_\Gamma z\Psi(1 - zw)\, \mathcal{Q}\left(\frac{z}{w}\right)\, \mathrm{d}w \\ &\quad - \frac{\gamma\, z}{2\pi i} \int_\Gamma \mathcal{Q}\left(\frac{z}{w}\right)\, \mathrm{d}w. \end{aligned}$$

The argument of the integral in the second summand has a power series with only even powers of w so the integral itself must vanish. We therefore have

$$(q \circledast \Phi)(z^2) = \frac{z}{2\pi i} \int_\Gamma \Psi(1 - zw)\, q\left(\frac{z}{w}\right) \mathrm{d}w$$

$\square$

Remark 4.3.19. In [46] it has been shown that the radius of convergence of $\mathcal{Q}$ is $\frac{4}{\pi} - 1$. Since $\zeta(m) \longrightarrow 1$ as $m \longrightarrow \infty$, it follows that the radius of convergence of $R_{Z(2\pi)}$ is also $\frac{4}{\pi} - 1$. It also follows that the R-transform of each $\xi_n \otimes \eta_n$ extends to a Pick function on $(1 - \frac{4}{\pi}, \frac{4}{\pi}, 1)$, see Section 4.4 below. Hence by Theorem 4.4.7 the law of $\xi_n \otimes \eta_n$ is $\boxplus$-infinitely divisible. Since free infinite divisibility is preserved by free linear combinations and weak limits, it follows that $Z(2\pi)$ is also $\boxplus$-infinitely divisible.

Unfortunately it seems that there is no explicit formula for $\mathcal{Q}$. It is therefore not apparent how a similar analysis to that for the square norm could be applied in order to obtain further details about the distribution of $Z(2\pi)$.

4.4 Lévy Area of the Free Brownian Bridge

In this section we use the Lévy representation

$$\beta(t) = \sum_{n=1}^{\infty} \frac{\cos(nt) - 1}{n\sqrt{\pi}}\, \xi_n + \sum_{n=1}^{\infty} \frac{\sin(nt)}{n\sqrt{\pi}}\, \eta_n \tag{4.4.1}$$

of the free Brownian bridge to compute the distribution of the free analogue of the classical Lévy area process defined by

$$\mathcal{L}(t) = \frac{i}{2} \int_0^t [\beta(s), \mathrm{d}\beta(s)] = \frac{i}{2} \int_0^t (\beta(s)\mathrm{d}\beta(s) - \mathrm{d}\beta(s)\beta(s))\,. \tag{4.4.2}$$

When β is a two-dimensional commutative Brownian motion this is very similar to the object studied by LÉVY [63]. By standard properties of the non-commutative integral [23] and self-adjointness of β we have

$$\int_0^t \beta(s)\mathrm{d}\beta(s) = \left(\int_0^t \mathrm{d}\beta(s)\beta(s)\right)^* .$$

A straightforward calculation yields that the left hand side equals, for $t = 2\pi$,

$$\int_0^{2\pi} \beta(s)\mathrm{d}\beta(s) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\xi_n \eta_n - \eta_n \xi_n\right) \tag{4.4.3}$$

which is easily seen to be anti-self-adjoint. This is the reason for the factor of i in (4.4.2): multiplying an anti-self-adjoint operator by i yields a self-adjoint random variable whose distribution is therefore supported in $\mathbb{R}$. Thus $\mathcal{L} := \mathcal{L}(2\pi)$ is equal to either side of (4.4.3) multiplied by i.

The summands are *commutators* of free semicircular random variables. Commutators have been studied by NICA–SPEICHER [82], where the semicircle distribution is discussed in Example 1.5(2). If $c_n = i\left(\xi_n \eta_n - \eta_n \xi_n\right)$, then the support of μ_{c_n} is $[-r, r]$ where $r = \sqrt{\frac{11+5\sqrt{5}}{2}}$ and

$$R_{c_n}(z) = \frac{2z}{1-z^2} = 2\sum_{m=1}^{\infty} z^{2m-1}. \tag{4.4.4}$$

From this we can now compute the R-transform of the classical Lévy area. Let that function be denoted $R_{\mathcal{L}}$ then

$$\begin{aligned} R_{\mathcal{L}} &= \sum_{n=1}^{\infty} \frac{1}{n} R_{c_n}\left(\frac{z}{n}\right) = \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2} \\ &= \frac{1}{z} - \pi\cot(\pi z). \end{aligned} \tag{4.4.5}$$

We can deduce the free cumulants of $\mathcal{L}$, either from the Taylor series of (4.4.5) or by calculating

$$R_{\mathcal{L}} = \sum_{n=1}^{\infty} \frac{2}{n} \sum_{m=1}^{\infty} \left(\frac{z}{n}\right)^{2m-1} = \sum_{m=1}^{\infty} 2 \left(\sum_{n=1}^{\infty} n^{-2m}\right) z^{2m-1}$$
$$= \sum_{m=1} 2\zeta(2m) z^{2m-1}$$

where the interchanging of the infinite sums is justified by absolute convergence. The free cumulants of $\mathcal{L}$are therefore given by

$$k_m(\mathcal{L}) = \begin{cases} 2\zeta(m) & \text{if } m \text{ is even} \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.6}$$

Free infinite divisibility is characterised by an analytic property of the R-transform. An analytic function $f\colon \mathbb{C}^+ \longrightarrow \mathbb{C}^+$ is called a *Pick function.* For $a, b \in \mathbb{R}$ with $a < b$ we denote by $\mathcal{P}(a, b)$ the set of Pick functions f which have an analytic continuation $g\colon \mathbb{C} \setminus \mathbb{R} \cup (a, b) \longrightarrow \mathbb{C}$ such that $g(\overline{z}) = \overline{g(z)}$. The following result is Theorem 3.3.6 of Hiai–Petz [55]:

Theorem 4.4.7. *A compactly supported probability measure μ is $\boxplus$-infinitely divisible if and only if its R-transform extends to a Pick function in $\mathcal{P}(-\epsilon, \epsilon)$ for some $\epsilon > 0$.*

It is easy to see that the common R-transform of the c_n extends to a Pick function in $\mathcal{P}(-1, 1)$. Therefore each c_n is $\boxplus$-infinitely divisible.

Corollary 4.4.8. *The distribution of $\mathcal{L}$ is $\boxplus$-infinitely divisible.*

As in Section 4.2 we can use free infinite divisibility together with the analytic properties of the R-transform and the formula for the maximum of the support from Chapter 3 to describe further the distribution in question.

The law of $\mathcal{L}$ is symmetric and therefore has vanishing odd free cumulants. So we now need to apply Theorem 3.5.9 (rather than Theorem 3.5.4) as for the square norm.

The inverse of the Cauchy transform of $\mathcal{L}$ is given by

$$K_{\mathcal{L}} = R_{\mathcal{L}} + \frac{1}{z} = \frac{2}{z} - \pi\cot(\pi z).$$

Similarly to the situation in Section 4.2.2 there exists, for every $t \in (\pi, 2\pi)$, unique $r(t) > 0$ such that $\mathfrak{Im}\left[K_{\mathcal{L}}\left(r(t)e^{it}\right)\right] = 0$ and

$$\frac{\partial}{\partial z}\mathfrak{Im}\left[K_{\mathcal{L}}(z)\right]\bigg|_{z=r(t)e^{it}} \neq 0 \qquad \forall t \in (\pi, 2\pi). \tag{4.4.9}$$

We obtain the following characterisation of the distribution of $\mathcal{L}$:

Proposition 4.4.10. *The non-commutative random variable $\mathcal{L}$ is distributed according to $\mu_{\mathcal{L}}(\mathrm{d}t) = \Phi_{\mathcal{L}}(t)\mathbf{1}_{[-\rho_{\mathcal{L}},\rho_{\mathcal{L}}]}\,\mathrm{d}t$ where $\Phi_{\mathcal{L}}(x) = -\frac{1}{\pi}r(\tau_x)\sin(\tau_x)$ and τ_x is the unique solution on $(\pi, 2\pi)$ to*

$$\frac{2}{r(\tau_x)\,e^{i\tau_x}} - \pi\cot\left(\pi r(\tau_x)\,e^{i\tau_x}\right) = x. \tag{4.4.11}$$

for every $x \in (-\rho_{\mathcal{L}}, \rho_{\mathcal{L}})$. The number $\rho_{\mathcal{L}}$ is given by

$$\rho_{\mathcal{L}} = \frac{m_*\pi}{\sqrt{m_*^2 - 2}} \tag{4.4.12}$$

where m_* is the unique solution on $(\sqrt{2}, \infty)$ of

$$m - 2 = \sqrt{m^2-2}\,\cot\left(\frac{\sqrt{m^2-2}}{m-1}\right). \tag{4.4.13}$$

A sketch of the density of $\mu_{\mathcal{L}}$ is given below.

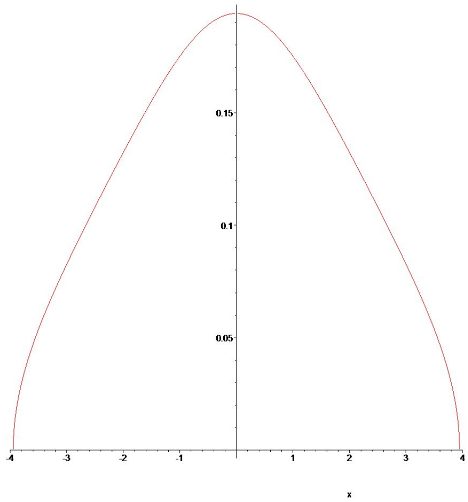

Figure 4.4: *
Density of the free Lévy area

Proof of Proposition 4.4.10. The law $\mu_{\mathcal{L}}$ of $\mathcal{L}$ is symmetric about 0. Together with the analytic arguments of Section 4.2.2, suitably modified, this implies the existence of $\rho_{\mathcal{L}} > 0$ such that the density $\Phi_{\mathcal{L}}$ of $\mu_{\mathcal{L}}$ is smooth, positive on $(-\rho_{\mathcal{L}}, \rho_{\mathcal{L}})$ and zero everywhere else. The function $\Phi_{\mathcal{L}}$ is given by $\Phi_{\mathcal{L}}(x) = -\frac{1}{\pi} r(\tau_x)\sin(\tau_x)$ where τ_x is characterised by (4.4.11).

For the remainder of the statement we apply Theorem 3.5.9. Only the free cumulants of even order are nonzero, so that the set L from Theorem 3.5.9 is given by $\{2n : n \in \mathbb{N}\}$. Otherwise the calculations are very similar to those in the proof of Proposition 4.2.11: we apply the methods of Lagrange multipliers and deduce that the supremum on the right-hand side of (3.5.10) is attained by a unique maximiser which is characterised by equation (4.4.13). The argument of the supremum evaluated at this maximiser yields the right edge of the support, and is given by (4.4.12). This completes the proof of the proposition. □

Remark 4.4.14. As for the square norm (cf. Remark 4.2.13), one could use the conformal mapping approach to find another implicit characterisation for the edges of spt $\mu_{\mathcal{L}}$.

Chapter 5

Analogues of Reflected Brownian Motion

In this chapter we study a generalisation of reflected Brownian motion introduced in Section 2.4. Rather than giving the process a singular drift whenever it hits one of the faces, we now impose a continuous drift. Its magnitude depends, via a potential U, on how far the process is away from being inside G.

5.1 Generalised RBM

For $k, d \in \mathbb{N}$ let $\{n_1, \ldots, n_k\}$ be a set of unit vectors spanning the whole of $\mathbb{R}^d$ and $q_j \in \mathbb{R}^d$ such that $q_j \cdot n_j = 0$ for all j. We denote by N, Q the $k \times d$ matrices whose j^{th} rows are given by n_j, q_j respectively and the polyhedral domain $G \subset \mathbb{R}^d$ by

$$G = \left\{x \in \mathbb{R}^d \colon n_j \cdot x \geq b_j \ \forall\, j \in \underline{k}\right\}$$

for some $b_1, \ldots, b_k \in \mathbb{R}$. We assume throughout that $G \neq 0$. For each $j \in \underline{k}$ we set $v_j := n_j + q_j$.

Let B be a Brownian motion with drift $-\mu$ and general covariance matrix $A = (a_{jk})_{j,k} = \sigma\sigma^T$, such that $a_{jj} = \alpha > 0$ for any j. Denote the generator of B by

$\mathfrak{a}$. We will assume throughout that the n_j, q_j satisfy the (modified) *skew-symmetry condition*:

$$n_j \cdot q_r + n_r \cdot q_j = \frac{2a_{rj}}{\alpha} \qquad \forall\, r \neq j. \tag{5.1.1}$$

Our focus lies on the cases when either $\mathfrak{a} = \frac{1}{2}\Delta - \mu\cdot\nabla$ (i.e. the covariance matrix of B is the identity), or $d = k$ and $N = I$ (i.e. the domain is an orthant). This corresponds to a generalisation of the processes introduced by Harrison–Williams [53, 130] and Harrison–Reiman [52] respectively. Note that in the former case (5.1.1) reduces to the skew-symmetry condition from [53, 130], cf. (2.4.1).

Let further U be a twice continously differentiable function, on which we make the following assumption.

Assumption 5.1.2. The continuously differentiable function $U\colon \mathbb{R} \longrightarrow \mathbb{R}$ is sufficiently regular to ensure that the second-order differential operator $\mathfrak{G}_U$ defined by

$$\mathfrak{G}_U = \mathfrak{a} - \mu \cdot \nabla + \sum_{r=1}^{k} U'\left(n_r \cdot x - b_r\right) v_r \cdot \nabla \tag{5.1.3}$$

is the infinitesimal generator of a diffusion process with continuous sample paths in $\mathbb{R}^d$. Moreover, if ρ is a smooth density on $\mathbb{R}^d$ satisfying $\mathfrak{G}_U^*\rho = 0$, then $\rho(x)\,\mathrm{d}x$ is an invariant measure for this diffusion. Here $\mathfrak{G}_U^*$ denotes the formal adjoint of $\mathfrak{G}_U$.

Definition 5.1.4. Let the function U satisfy Assumption 5.1.2. The $\mathbb{R}^d$-valued diffusion with infinitesimal generator $\mathfrak{G}_U$ is called *generalised reflected Brownian motion (GRBM)* corresponding to the potential U and the data (N, Q, μ, A).

The choice $U(x) = -e^{-x}$ satisfies Assumption 5.1.2. We refer to the process with this choice of U as the *exponentially reflected Brownian motion (ERBM)* corresponding to (N, Q, μ, A).

We will consider the cases when either $\mathfrak{a} = \frac{1}{2}\Delta$ (i.e. the covariance matrix of B is the identity), or $d = k$ and $N = I$ (i.e. the domain is an orthant). This corresponds to a generalisation of the results by Harrison–Williams [53, 130] and Harrison–Reiman [52] respectively.

Remark 5.1.5. Apart from the requirement that the n_j contains a basis of $\mathbb{R}^d$ one could allow $k < d$, that is fewer half-spaces than the dimension we are in. We can then decouple the 'superfluous' dimensions as follows. Denote by $E_1 \cong \mathbb{R}^k$ the span of $\{n_1, \ldots, n_k\}$ and by E_2 its orthogonal complement in $\mathbb{R}^d$. Let further P_j be orthogonal projection from $\mathbb{R}^d$ onto E_j. By orthogonal invariance of Brownian motion it follows that $P_1(X)$ and $P_2(X)$ are independent. Further $P_2(X)$ is just a standard $(d-k)$-dimensional Brownian motion, whereas $P_2(X)$ is generalised RBM in $\mathbb{R}^k$ with the data $(P_1(N), P_1(Q), P_1(\mu), I)$.

With this in mind we will assume throughout that $k \geq d$.

5.2 Main Results

Let us state the main abstract results, the proofs of which can be found in Section 5.4.

5.2.1 GRBM in a General Domain

We first consider the Harrison–Williams setting. Our main result is that, under the skew-symmetry condition, generalised reflected Brownian motion has an invariant measure in a certain product form.

Theorem 5.2.1. *Suppose that U satisfies Assumption 5.1.2 and that the n_j, q_j satisfy the skew-symmetry condition (5.1.1). Then the $\mathbb{R}^d$-valued diffusion with generator*

$$\mathfrak{G}_U = \frac{1}{2}\Delta + \left(\sum_{r=1}^{k} U'\left(n_r \cdot x - b_r\right) v_r - \mu\right) \cdot \nabla$$

has as invariant measure of the form $\nu_U(\mathrm{d}x) = p_U(x)\,\mathrm{d}x$ *with*

$$p_U(x) = \exp\left\{2\left(\sum_{r=1}^{k} U\left(n_r \cdot x - b_r\right) - \gamma(\mu)\cdot x\right)\right\}\mathrm{d}x \tag{5.2.2}$$

where $\gamma(\mu)$ *is given by*

$$\gamma(\mu) = \left(I - \overline{N}^{-1}\overline{Q}\right)^{-1}\mu. \tag{5.2.3}$$

Here, $\overline{N}$ *is an invertible* $d \times d$ *submatrix of* N *and* $\overline{Q}$ *is the corresponding submatrix of* Q.

Remark 5.2.4. The existence of an invertible submatrix $\overline{N}$ of N was assumed. By the remarks after equation (1.7) in [53] (p. 463) the matrix $(I - \overline{N}^{-1}\overline{Q})$, and hence $\gamma(\mu)$, is independent of the choice of $\overline{N}$, provided the skew-symmetry condition (5.1.1) holds. Further [53, (4.7), (7.13)] we have $|\gamma(\mu)|^2 = \gamma\cdot\mu$.

Applied to the special case $U(x) = -e^{-x}$, Theorem 5.2.1 gives the invariant measure for exponentially reflected Brownian motio.

Corollary 5.2.5. *Suppose that the skew-symmetry condition (5.1.1) holds, then the exponentially reflecting Brownian motion corresponding to* (N, Q, μ, I) *has an invariant measure* ν *in product form. More precisely,* ν *is absolutely continuous with respect to Lebesgue measure with density*

$$\nu(\mathrm{d}x) = \frac{1}{Z}\exp\left\{-2\left(\gamma(\mu)\cdot x + \sum_{j=1}^{k} e^{b_j - n_j\cdot x}\right)\right\}.$$

Remark 5.2.6. When $d = k$ then ν can be realised as the distribution of a $\mathbb{R}^d$-valued random variable $X(\infty)$ such that

$$(n_1\cdot X(\infty), \ldots, n_d\cdot X(\infty)) \overset{(\mathrm{d})}{=} \left(-\log\left(\frac{\gamma_1}{2}\right), \ldots, -\log\left(\frac{\gamma_d}{2}\right)\right)$$

where the γ_j are independent gamma random variables with parameters q_j and the vector θ is given by

$$\theta = 2(N+Q)^{-T}\mu.$$

Remark 5.2.7. Let $\beta > 0$ and set $U_\beta(x) = -\frac{1}{\beta}e^{-\beta x}$ for $\beta > 0$. The diffusion with generator $\mathfrak{G}_{U_\beta}$ should converge, as $\beta \to \infty$ to the Harrison–Williams reflected Brownian motion. Moreover (cf. [33], section 4.1) the log-gamma random variables converge to the exponential distribution, and we recover the main result of [130]. In this sense our results can be considered as a generalisation of those of [53, 130].

5.2.2 General Covariance

We now turn to exponential Brownian motion in an orthant, driven by a d-Brownian motion with drift $-\mu$ that is allowed to have a general, possibly singular covariance. It can be realised as $B(t) = \sigma\beta(t) - \mu t$ for a standard Brownian motion β with no drift, possibly of a different dimension, and a matrix σ, which is generally rectangular. The covariance matrix $A = (a_{jk}) = \sigma\sigma^T$ of B is assumed to have the same entry $\alpha > 0$ on all its diagonal entries (or, equivalently, the rows of the rectangular matrix σ all have the same length $\sqrt{\alpha}$).

In this case the generator of the generalised RBM is given by

$$\mathfrak{G}_U = \frac{1}{2}\sum_{j,l} a_{j,l}\partial_{x_j}\partial_{x_k} + \sum_{j=1}^{d}\left[\sum_{r=1}^{d} q_{rj}U'(x_r) + U'(x_j) - \mu_j\right]\partial_{x_j}.$$

Applied to this setting the modified skew-symmetry condition (5.1.1) reads

$$q_{jr} + q_{rj} = \frac{2a_{rj}}{\alpha} \tag{5.2.8}$$

Theorem 5.2.9. *Suppose that $U\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Assumption 5.1.2 and that for* *all $j \neq r$ the* modified skew-symmetry condition (5.2.8) *holds. Then the GRBM in an orthant, corresponding to (I, Q, μ, A) and the potential U has an invariant measure whose density with respect to Lebesgue measure is given by*

$$p_U(x) = \frac{1}{Z} \exp\left\{ 2 \left[\sum_{j=1}^{d} U(x_j) - \sqrt{\alpha}\,(2A - \alpha (I+Q))^{-1} \mu \cdot x \right] \right\}. \qquad (5.2.10)$$

5.3 Examples

5.3.1 One-Dimensional ERBM and Dufresne's Identity

As a warm-up let us consider one-dimensional exponentially RBM. Here we can use a particular realisation of the process and Dufresne's identity.

In this situation $n = 1$ and $q = 0$. All conditions, including skew-symmetry (5.1.1), are satisfied. Let further $\mu > 0$. The generator of X in this simple case is given by

$$\mathfrak{G}_1^{(\mu)} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(e^{-x} - \mu\right)\frac{\mathrm{d}}{\mathrm{d}x}.$$

By Itô's formula and stochastic integration by parts [57, 98] the process X given by

$$X(t) = \log \int_0^t e^{B^{(\mu)}(s) - B^{(\mu)}(t)}\,\mathrm{d}s$$

is a diffusion with generator $\mathfrak{G}_1^{(\mu)}$. The invariant measure of X is that of $\eta = \log(\xi)$ where $\xi \overset{(\mathrm{d})}{=} 4A_\infty^{(2\mu)}$ and the process $A^{(\mu)}$ is defined by [41, 42]

$$A_t^{(\mu)} = \int_0^t e^{2(B(s) - \mu s)}\,\mathrm{d}s.$$

Recall Dufresne's identity [41, Corollary 4].

Proposition 5.3.1. *Let $\mu > 0$, then*

$$\left(2A_\infty^{(\mu)}\right)^{-1} \overset{(\mathrm{d})}{=} \gamma_\mu$$

where γ_μ has the Gamma distribution with parameter μ.

Hence $\xi \overset{(\mathrm{d})}{=} \frac{2}{\gamma_{2\mu}}$ and so the invariant distribution of the process X is realised by

$$\eta = -\log \xi^{-1} \overset{(\mathrm{d})}{=} -\log\left(\frac{\gamma_{2\mu}}{2}\right).$$

which recovers our result in this simple example. Let us also note that, replacing the potential e^{-x} by $\frac{1}{\beta}e^{-\beta x}$ (as in Remark 5.2.7) it can be verified directly that both Harrison–Williams RBM and the exponential distribution appear in the scaling limit as $\beta \to \infty$.

5.3.2 RBM in a Weyl Chamber

Reflected Brownian motion in the *Weyl chamber*

$$\Omega = \left\{x \in \mathbb{R}^d \colon x_1 > x_2 > \ldots > x_d\right\},$$

with normal reflection is a realisation of *Brownian motion with rank-dependent drift*, studied by PAL–PITMAN [92]. This is defined by taking d standard Brownian motions $X_1, \ldots, X_d$ with increasing re-ordering $X_{(1)}, \ldots, X_{(d)}$ and drift μ with $-\mu \in \Omega$ such that at each time t the process $X_{(j)}$ has drift μ_j. It can also be viewed as a Doob transform of the *Delta-Bose gas*, see PROLHAC–SPOHN [95]. In our generalised setting the generator is given by

$$\begin{aligned}\mathfrak{G}_U = \frac{1}{2}\Delta + \sum_{r=2}^{d-1}\left[U'(x_r - x_{r+1}) - U'(x_{r-1} - x_r) - \mu_r\right]\partial_{x_r}\\ + \left[U'(x_1 - x_2) - \mu_1\right]\partial_{x_1} - \left[U'(x_{d-1} - x_d)\mu_d\right]\partial_{x_d}.\end{aligned}$$

Because of normal reflection the skew-symmetry condition is automatic and hence the system has a stationary distribution in product form with density

$$p_\mu(x) = \frac{1}{Z_\mu} \exp\left\{ 2 \left(\sum_{j=1}^{d-1} U\left(x_{j+1} - x_j\right) - \mu \cdot x \right) \right\}.$$

5.3.3 Examples Motivated by Queueing Theory

As mentioned in the introduction, the primary motivation of studying reflected Brownian motion in a polyhedral domain came from queueing theory. We present here some examples of exponential reflected Brownian motion that correspond to networks consisting of several instances of the *generalised Brownian queue* introduced in [89]. For background on queueing theory we refer to Section 2.4.3 and the references given there.

Example 5.3.2. Let B_1, B_2, B_3 be three Brownian motions with drifts ν_j such that for $j \in \{1,2\}$ we have $\mu_j := \nu_{j+1} - \nu_j > 0$. Let X_1, X_2 be the queue-length processes of two generalised Brownian queues in tandem, that is

$$\begin{aligned} X_1(t) &= \log \int_0^t \exp\left\{B_1(t) - B_1(s) - B_2(t) + B_1(s)\right\} \mathrm{d}t \\ D(t) &= B_1(t) + X_1(0) - X_1(t) \\ X_2(t) &= \log \int_0^t \exp\left\{D(t) - D(s) + B_3(s) - B_3(t)\right\} \mathrm{d}t. \end{aligned}$$

This corresponds to ERBM in an orthant with reflection and covariance matrices given by

$$Q = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

i.e. the scaling parameter is $\alpha = 2$. It is straightforward that these choices satisfy the skew-symmetry condition (5.2.8), so we get a stationary distribution in product form. More precisely, by (5.2.10) the stationary distribution has density

$$p(x,y) = \frac{1}{Z}\exp\left\{-2\left[\frac{\mu_1+\mu_2}{\sqrt{2}}\,x + \frac{\mu_2}{\sqrt{2}}\,y + e^{-x} + e^{-y}\right]\right\}.$$

Example 5.3.3. Let us consider another example motivated by queueing theory, namely a system of two generalised Brownian queues set up so that the departures from the first queue become the arrivals in the second, and the unused service process of the first is the service process of the second queue. The corresponding ERBM is defined by the data

$$Q = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

so we don't need to re-scale. It is straightforward to check the skew-symmetry condition:

$$q_{12} + q_{21} = -2 = 2a_{12}$$

and hence we get a product-form stationary distribution, which has density

$$p(x,y) = \frac{1}{Z}\exp\left\{-2\left[(\mu_1+2\mu_2)\,x + \mu_2 y + e^{-x} + e^{-y}\right]\right\}.$$

Example 5.3.4. We can take this one step further: take three queues such that the arrivals and services of each subsequent queue are given by the departures and unused services of the previous one. Then

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 2 & -2 & 0 \end{pmatrix}, \quad AA^T = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

and again we are in the right scale for unit vectors. Similarly to the two-dimensional case we can check skew-symmetry, so we get another stationary distribution in product form, this time with density given by

$$p(x,y,z) = \frac{1}{Z} \exp\left\{-2\left[(\mu_1 + 2\mu_2 + 2\mu_3)\, x + (\mu_2 + 2\mu_3)\, y + \mu_3 z + e^{-x} + e^{-y} + e^{-z}\right]\right\}$$

The two previous examples generalise as follows. Fix $d \in \mathbb{N}$ and a sequence $\alpha_1, \alpha_2, \ldots \in \mathbb{R}^d$ such that $\alpha_j \cdot \alpha_j = 2$ for all j. Let η be a Brownian motion in $\mathbb{R}^d$ with drift μ and define $y_1, \ldots, y_d$ by the system of SDEs

$$\mathrm{d}y_k(t) = \begin{cases} \mathrm{d}\left(\alpha_1 \cdot \eta(t)\right) + e^{-y_1(t)}\, \mathrm{d}t & \text{if } k = 1 \\ \mathrm{d}\left(\alpha_k \cdot \eta(t)\right) + \left(\sum_{j=1}^{k-1} (\alpha_k \cdot \alpha_j)\; e^{-y_j(t)} + e^{-y_k(t)}\right) \mathrm{d}t & \text{if } k > 1 \end{cases}$$

A special case of this process is discussed in [86], section 7.3. See also [21, 22].

Proposition 5.3.5. *The d-dimensional diffusion y has a stationary distribution π in product form. The marginals are of the form $-\log(\gamma_{\theta_k}/2)$ where the γ_{θ_k} are gamma random variables with parameters θ_k. The vector θ is given by $\theta = \theta(\mu) = \frac{1}{\sqrt{2}} C\mu$ where,*

$$C_{jk} = \begin{cases} 0 & \textit{if } j > k \\ 1 & \textit{if } j = k \\ -\alpha_j \cdot \alpha_{j+1} & \textit{if } k = j+1 \\ -s_{\alpha_{k-1}} \ldots s_{\alpha_{j+1}} (\alpha_j) \cdot \alpha_k & \textit{otherwise} \end{cases}$$

Proof. The process y is RBM in a d-dimensional orthant, driven by the process η

given by

$$\eta(t) = \begin{pmatrix} \alpha_1^T \\ \vdots \\ \alpha_d^T \end{pmatrix} \beta(t)$$

with β a standard Brownian motion. The covariance matrix of η is therefore given by $a_{jk} = \alpha_j \cdot \alpha_k$, while the reflection matrix is $Q = (q_{jk})$ where

$$q_{jk} = \begin{cases} 0 & \text{if } j \geq k \\ \alpha_j \cdot \alpha_k & \text{otherwise.} \end{cases}$$

We assumed that $\alpha_j \cdot \alpha_j = 2$ so the scaling parameter is $\alpha = 2$. We can check that the skew symmetry condition holds: whenever $r \neq j$ we have

$$q_{jr} + q_{rj} = \alpha_j \cdot \alpha_r = \frac{2a_{jr}}{\alpha}$$

since one of the summands on the left equals the right-hand side and the other vanishes. So we can apply Theorem 5.2.9 to complete the proof. □

5.4 Proofs of the Main Results

In this final section we prove Theorems 5.2.1 and 5.2.9. By Assumption 5.1.2 we only need to show that $\mathfrak{G}^* p = 0$ where $\mathfrak{G}^*$ is the formal adjoint of the generator $\mathfrak{G}$ and p is as in (5.2.2) and (5.2.10) respectively.

5.4.1 General Polyhedral Domain

The generator of GRBM in a general polyhedral domain, driven by a standard Brownian motion, is of the form $\mathfrak{G} = \frac{1}{2}\Delta + \mathbf{\Omega} \cdot \nabla$ where

$$\mathbf{\Omega}(x) = \sum_{r=1}^{k} U'\left(n_r \cdot x - b_r\right) v_r - \mu$$

Using integration by parts the formal adjoint of $\mathfrak{G}$ is given by

$$\mathfrak{G}^* = \frac{1}{2}\Delta - \mathbf{\Omega} \cdot \nabla - \nabla \cdot \mathbf{\Omega}.$$

We remark that p has the form $p(x) = \exp\{W(x)\}$ where

$$W(x) = 2\left[\sum_{r=1}^{k} U\left(n_r \cdot x - b_r\right) - \gamma \cdot x\right].$$

Let us remark that

$$\mathbf{\Omega}(x) = \frac{1}{2}\nabla W(x) + (\gamma - \mu) + \sum_{j=1}^{k} U'\left(n_r \cdot x - b_r\right) q_r.$$

Because $q_j \cdot n_j = 0$ for all j we have $\nabla \cdot \mathbf{\Omega} = \frac{1}{2}\Delta W$. Further $\nabla p = p\nabla W$ and therefore $\Delta p = \left(\Delta W + |\nabla W|^2\right) p$. Hence it follows that

$$\begin{aligned}
\mathfrak{G}^* p &= \frac{1}{2}\Delta p - \left[\frac{1}{2}\nabla W + (\gamma - \mu) + \sum_{j=1}^{k} U'\left(n_r \cdot x - b_r\right) q_r\right] \cdot \nabla p - \frac{1}{2} W p \\
&= \frac{1}{2}\left[\Delta W + \frac{1}{2}|\nabla W|^2\right] p - \frac{1}{2}|\nabla W|^2 p + (\gamma - \mu) \cdot \nabla W p \\
&\quad - \sum_{r=1}^{k} U'\left(n_r \cdot x - b_r\right) q_r \cdot \nabla p - \frac{1}{2}\Delta W p.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\frac{\mathfrak{G}^* p(x)}{p(x)} &= (\mu-\gamma)\cdot\nabla W(x) - \sum_{r=1}^{k} U\left(n_r\cdot x - b_r\right) q_r \cdot \nabla W(x) \\
&= 2\sum_{j=1}^{k} U'\left(n_j\cdot x - b_j\right)(\mu-\gamma)\cdot n_j - 2\left(\mu-\gamma\right)\cdot\gamma \\
&\qquad - \sum_{r,s} U'\left(n_r\cdot x - b_r\right) U'\left(n_s\cdot x - b_s\right) n_r\cdot q_s - 2\sum_{r=1}^{k} U'\left(n_r\cdot x - b_r\right)\gamma\cdot q_s \\
&= 2\sum_{r=1}^{k} U'\left(n_r\cdot x - b_r\right)\left[N\mu - (N-Q)\gamma\right]_r .
\end{aligned}
$$

where we have used the skew-symmetry condition and $[\boldsymbol{y}]_r$ denotes the r^{th} entry of a vector $\boldsymbol{y}$. The fact that the last line is equal to zero follows from the fact that $\overline{N}\mu = (\overline{N} - \overline{Q})$ for *any* choice of invertible submatrix $\overline{N}$ of N (and corresponding submatrix $\overline{Q}$ of Q) and that each row of N must occur in at least one invertible submatrix, because of Lemma 5.4.1 below.

In the proof above we have used the following simple lemma from Linear Algebra.

Lemma 5.4.1. *Let $k > d$ and $\mathcal{E} = \{y_1, \ldots, y_k\} \subset \mathbb{R}^d$ a set of vectors whose span is the whole of $\mathbb{R}^d$. If there exists $r \in \underline{k}$ such that any collection of d elements of $\mathcal{E}$ containing y_r is linearly dependent then $y_r = 0$.*

Proof. The assumption that $\mathcal{E}$ spans $\mathbb{R}^d$ is equivalent to $\mathcal{E}$ containing a basis. By re-ordering we may therefore assume that $\{y_1, \ldots, y_d\}$ is such a basis. If now the set $\{y_1, \ldots, \widehat{y}_j, \ldots, y_{d+1}\}$ (meaning the vector y_r is left out) is linearly dependent for $r \in \underline{d}$ then y_{d+1} is in the span of $\{y_1, \ldots, \widehat{y}_r, \ldots y_d\}$. By assumption this is true for *every* $r \in \underline{d}$, so

$$
y_{d+1} \in \bigcap_{r=1}^{d} \operatorname{span}\{y_1, \ldots, \widehat{y}_r, \ldots, y_d\} = \{0\}
$$

which completes the proof. $\square$

5.4.2 Orthant with General Covariance

Finally we turn to proving Theorem 5.2.9, giving the invariant measure for generalised RBM in an orthant, driven by a Brownian motion with general covariance, whose generator is denoted by $\mathfrak{a}$.

The generator of this process is now of the form

$$\mathfrak{G} = \frac{1}{2}\nabla \cdot \mathfrak{a}\nabla + \mathbf{\Omega} \cdot \nabla$$

where $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega}(x) = \sum_{j=1}^{d} U'(n_j \cdot x)(e_j + q_j) - \mu.$$

The formal adjoint of $\mathfrak{G}$ is given by $\mathfrak{G}^* = \frac{1}{2}\nabla \cdot \mathfrak{a}\nabla - \mathbf{\Omega}\cdot\nabla - \nabla\dot{\mathbf{\Omega}}$, and we need to prove that $\mathfrak{G}^* p = 0$ where $p(x) = \exp\{W(x)\}$,

$$W(x) = 2\left(\sum_{j=1}^{d} U(x_j) - \delta(\mu)\cdot x\right).$$

and $\delta(\mu) = (2\mathfrak{a} - I - Q)^{-1}\mu$. Noting that

$$\mathbf{\Omega}(x) = \frac{1}{2}\nabla W(x) + \sum_{j=1}^{d} U'(x_j)\, q_j + \delta(\mu) - \mu$$

and hence $\nabla\cdot\Omega = \frac{1}{2}\Delta W$ it follows that

$$\begin{aligned}\mathfrak{G}^* p = \frac{1}{2}\left(\nabla\cdot\mathfrak{a}\nabla + \nabla W\cdot\mathfrak{a}\nabla W\right)p - \left[\left(\frac{1}{2}\nabla W + \sum_{j=1}^{d} U'(n_j\cdot x)\, q_j\right)\cdot\nabla W\right]p \\ + (\mu - \delta(\mu))\cdot\nabla W p - \frac{p}{2}\Delta W.\end{aligned}$$

Dividing by p and then using $\nabla W(x) = 2\left(\sum_j U'(n_j \cdot x)\, n_j - \delta(\mu)\right)$ we obtain

$$\begin{aligned}
\frac{\mathfrak{G}^* p(x)}{p(x)} &= \frac{1}{2} \nabla \cdot (\mathfrak{a} - I) \nabla W(x) + \frac{1}{2} \nabla W(x) \cdot \nabla (\mathfrak{a} - I) \nabla W(x) \\
&\quad - \sum_{j=1}^{d} U'(x_j)\, q_j \cdot \nabla W(x) + (\mu - \delta(\mu)) \cdot \nabla W(x) \\
&= \frac{1}{2} \sum_{j=1}^{d} \left[U''(x_j)\, (\mathfrak{a} - I)_{jj} \right] + 2 \sum_{j,r} U'(x_j)\, U'(x_r)\, (\mathfrak{a} - I)_{jr} \\
&\quad - 2 \sum_{j=1}^{d} U'(x_j) \left[\delta(\mu) \cdot (\mathfrak{a} - I)\, e_j + e_j \cdot (\mathfrak{a} - I)\, \delta(\mu) \right] + 2\delta(\mu) \cdot (\mathfrak{a} - I)\, \delta(\mu) \\
&\quad + 2 \sum_{j=1}^{d} U'(x_j) \left[q_j \cdot \delta(\mu) + (\mu_j - \delta(\mu)_j) \right] \\
&\quad - 2 \sum_{j,r} U'(x_j)\, U'(x_r)\, q_{jr} + 2\delta(\mu) \cdot (\delta(\mu) - \mu)
\end{aligned}$$

The first term vanishes because the diagonal entries of $\mathfrak{a}$ are all equal to 1. Using the fact that $\mathfrak{a}$ is symmetric we have

$$\begin{aligned}
\frac{\mathfrak{G}^* p(x)}{p(x)} &= 2 \sum_{j,r} U'(x_j)\, U'(x_r) \left[(\mathfrak{a} - I)_{jr} - q_{jr} \right] \\
&\quad + 2 \sum_{j=1}^{d} U'(x_j) \left[q_j \cdot \delta(\mu) + (\mu_j - \delta(\mu)_j) - 2e_j \cdot (\mathfrak{a} - I)\, \delta(\mu) \right] \\
&\quad + 2\delta(\mu) \cdot (\mathfrak{a} - I)\, \delta(\mu) + 2\delta(\mu) \cdot (\delta(\mu) - \mu)
\end{aligned}$$

The first line equals zero because of the skew-symmetry condition, whereas the other two lines vanish due to the definition of $\delta(\mu)$.

This completes the proof of Theorem 5.2.9. $\square$

Appendix A

Computations for the Square Norm

A.1 Unique Solution in Polar Co-Ordinates

Let β be a free Brownian bridge and put $\Gamma := \int_0^1 \beta^2(s)\,\mathrm{d}s$. Let R_Γ, K_Γ, G_Γ denote the R-, K-, Cauchy transform of Γ respectively. From the main paper we have

$$K_\Gamma(z) = \frac{3 - \sqrt{z}\cot\left(\sqrt{z}\right)}{2z}. \tag{A.1.1}$$

The purpose of this section is to establish

Theorem A.1.2. *For every $t \in (\pi, 2\pi)$ there exists unique $\rho_0(t) > 0$ such that*

$$\mathfrak{Im}\left(K\left(\rho_0(t)e^{it}\right)\right) = 0 \tag{A.1.3}$$

Let us first remark that

$$\begin{aligned}\mathfrak{Re}\left(\cot(R\,e^{i\theta})\right) &= \frac{\sin(R\cos(\theta))\,\cos(R\cos(\theta))}{\sin^2(R\cos(\theta)) + \sinh^2(R\sin(\theta))}\\ \mathfrak{Im}\left(\cot(R\,e^{i\theta})\right) &= -\frac{\sinh(R\sin(\theta))\,\cosh(R\sin(\theta))}{\sin^2(R\cos(\theta)) + \sinh^2(R\sin(\theta))}.\end{aligned}$$

Put $\gamma = \gamma(t) = \cos(t/2)$, $\sigma = \sigma(t) = \sin(t/2)$, and note that $t \longmapsto \gamma(t)$ and $t \longmapsto \sigma(t)$ are both bijections on $(\pi, 2\pi)$. Then

$$\begin{aligned}
\mathfrak{Im}K(r\,e^{it}) &= \frac{1}{2}\mathfrak{Im}\left(\frac{3\,e^{it}}{r}\right) - \left\{\frac{1}{2\sqrt{r}}\,\mathfrak{Im}\left(e^{it/2}\,\cot\left(\sqrt{r}\,e^{it/2}\right)\right)\right\}\\
&= \frac{-3\sin(t)}{2r} + \frac{\gamma}{\sqrt{r}}\,\frac{\sinh(\sigma\sqrt{r})\cosh(\sigma\sqrt{r})}{\sin^2(\gamma\sqrt{r}) + \sinh^2(\sigma\sqrt{r})}\\
&\qquad + \frac{\sigma}{\sqrt{r}}\,\frac{\sin(\gamma\sqrt{r})\cos(\gamma\sqrt{r})}{\sin^2(\gamma\sqrt{r}) + \sinh^2(\sigma\sqrt{r})} \qquad \text{(A.1.4)}\\
&= -\frac{3\sin(t)}{r} + \frac{\gamma\sinh(\sigma\sqrt{r})\cosh(\sigma\sqrt{r}) + \sigma\sin(\gamma\sqrt{r})\cos(\gamma\sqrt{r})}{\sin^2(\gamma\sqrt{r}) + \sinh^2(\sigma\sqrt{r})} \qquad \text{(A.1.5)}
\end{aligned}$$

Denote the right-hand side of (A.1.5) by $\mathcal{L}(r,t)$ and define $g_t(r) := 2r\mathcal{L}(r^2, t)$. Then, using the fact that $\sin(t) = 2\sin(t/2)\cos(t/2)$ we have

$$g_t(r) = -\frac{6\sigma\gamma}{r} + \frac{\sigma\sin(\gamma r)\cos(\gamma r) + \gamma\sinh(\sigma r)\cosh(\sigma r)}{\sin^2(\gamma\sqrt{r}) + \sinh^2(\sigma\sqrt{r})} \tag{A.1.6}$$

and clearly Theorem A.1.2 is proved if we establish the following result.

Theorem A.1.7. *For each $t \in (\pi, 2\pi)$ there exists unique $r = r_t > 0$ such that $g_t(r_t) = 0$. Moreover r_t has the property that $g_t'(r_t) < 0$.*

Our strategy of proof is as follows: fix $t \in (\pi, 2\pi)$. For convenience we will often suppress the suffix t and write $g(r), \gamma, \sigma$ etc. We know that

$$\lim_{r\to 0} g(r) = +\infty$$

so there must be $R_2 > 0$ sucht that $g(r) > 0\ \forall\, r \in (0, R_2)$. Our strategy is to prove the following:

(a) $\exists\, R_1 > 0\ \forall\, r \in (0, R_1)\colon g'(r) < 0$

(b) $\exists\, R_3 \in (0, R_1)\ \forall\, r > R_3\colon g(r) < 0$.

If this holds, then there must be a zero of g in $[R_2, R_3]$. Since g is decreasing on that interval it must be unique on that interval, label it $r_0(t)$. Since g is positive on $(0, R_2)$ and negative on (R_3, ∞), that $r_0(t)$ is in fact the only root of g on $(0, \infty)$.

We will be distinguishing between several cases. Since we will need it frequently in the subsequent sections we note here that

$$g'(r) = \frac{6\sigma\gamma\left(\sin^2(\gamma r) + \sinh^2(\sigma r)\right)^2 - r^2\sin(2\gamma r)\sinh(\sigma r)\cosh(\sigma r)}{r^2\left(\sin^2(\gamma r) + \sinh^2(\sigma r)\right)^2} \tag{A.1.8}$$

A.1.1 First Case: $t \in (\pi, \frac{3\pi}{2})$.

If $t \in \left(\pi, \frac{3\pi}{2}\right)$ then $\sigma \in (\frac{1}{\sqrt{2}}, 1)$ and $-\gamma \in (0, \frac{1}{\sqrt{2}})$. In this case it turns out that the function g is actually always decreasing:

Lemma A.1.9. *Under these assumptions on t (and hence σ and γ) we have $g'(r) < 0$ for all $r > 0$.*

Proof. We will split the proof of this result into two subcases as to whether $\sigma^4 \geq \frac{1}{3}$ or not.

(a) $\sigma \in (3^{-1/4}, 1]$. Denote the numerator of (A.1.8) by $h(r)$ so that

$$\begin{aligned} h(r) &= 6\sigma\gamma\left(\sin^2(\gamma r) + \sinh^2(\sigma r)\right)^2 - r^2\sin(2\gamma r)\sinh(\sigma r)\cosh(\sigma r) \quad \text{(A.1.10)} \\ &\leq 6\sigma\gamma\sinh^4(\sigma r) - 2\gamma r^3\sinh(\sigma r)\cosh(\sigma r) \\ &\leq \frac{2\gamma}{\sigma^3}\left(3\sigma^4\sinh^4(\sigma r) - (\sigma r)^3\sinh(\sigma r)\cosh(\sigma r)\right) \\ &\leq \frac{2\gamma}{\sigma^3}\sinh(\sigma r)\left(\sinh^3(\sigma r) - (\sigma r)^3\cosh(\sigma r)\right) < 0 \end{aligned}$$

since the function defined by $p(x) = \sinh^4(x) - x^3\cosh(x)$ is positive on $[0, \infty)$ and $\sigma\gamma\sinh(\sigma r) < 0$.

(b) $\sigma \in (2^{-1/2}, 3^{-1/4})$. For this range of σ we have $-3\sigma\gamma > \sqrt{2}$ and $\sigma > -\gamma > \frac{\sigma}{2}$. Retain notation for $h(r)$ as in (A.1.10). We split into sub-subcases according to whether σr is bigger or smaller than $\pi/4$.

(i) $\sigma r \leq \frac{\pi}{4}$. Then $-\gamma r, -2\gamma r$ are both less than $2\sigma r \leq \frac{\pi}{2}$ and since the sine function is increasing on $\left[0, \frac{\pi}{2}\right]$ we have

- $\sin^2\left(\gamma(r)\right) \geq \sin^2\left(\frac{\sigma r}{2}\right)$
- $\sin(-2\gamma r) \leq \sin(2\sigma r)$.

So, using $\sigma^{-1} \leq \sqrt{2}$ we have

$$h(r) \leq 2\left[\underbrace{3\sigma\gamma\left(\sin^2(\sigma r/2) + \sinh^2(\sigma r)\right)^2}_{-f_1(\sigma r)}\right.$$
$$\left. + \underbrace{(\sigma r)^2 \sin(-2\gamma r)\sinh(\sigma r)\cosh(\sigma r)}_{f_2(\sigma r)}\right]$$

We are done if we can show that $f_1^1(x) > f_2^2(x)$. Using the fact that $(3\sigma\gamma)^2 > 2$ and the identity $\cosh^2 = 1 + \sinh^2$ we obtain

$$\begin{aligned} f_1^2(x) &> 2\left(\sin^2(x/2) + \sinh^2(x)\right)^4 \\ &\geq 2\sinh^8(x) + 8\sin^2(x/2)\sinh^6(x) =: \widetilde{f}_1(x) \\ f_2^2(x) &= x^4\sin^2(x)\sinh^2(x) + x^4\sin^2(2x)\sinh^4(x) =: \widetilde{f}_2(x) \end{aligned}$$

and it is straightforward to check that $\widetilde{f}_1(x) > \widetilde{f}_2(x)$ if $x \in (0, \pi/4)$.

(ii) $\sigma r > \frac{\pi}{4}$. Then we can estimate even more crudely, still using $-3\sigma\gamma > \sqrt{2}$ and putting $x = \sigma r$:

$$\begin{aligned} h(r) &< 2\left[3\sin\gamma\sinh^4(\sigma r) + (\sigma r)^2\sinh(\sigma r)\cosh(\sigma r)\right] \\ &< 2\sinh(x)\left[\underbrace{-\sqrt{2}\sinh^3(x) + x^2\cosh(x)}_{P(x)}\right]. \end{aligned}$$

But $P(\pi/4) < 0$ by direct calculation and

$$P'(x) = -3\sqrt{2}\sinh^2(x)\cosh(x) + 2x\cosh(x) + x^2\sinh(x) < 0$$

for $x > \pi/4$ since then $\sqrt{2}\sinh(x) > 1$. Hence $P(x) < 0$ for $x > \pi/4$ and so $h(r) < 0$ for (ii) also.

This concludes (b) and hence the proof.

□

So it only remains to show that $g(a) < 0$ for some $a > 0$:

Lemma A.1.11. *We have $g(7/\sigma) < 0$.*

Proof. Note that

$$\begin{aligned} g(r) &< \frac{-6\sigma\gamma}{r} - \frac{\sigma\gamma r}{\sinh^2(\sigma r)} + \frac{\gamma}{4}\frac{\sinh(\sigma r)\cosh(\sigma r)}{1+\sinh^2(\sigma r)} \\ &< -\gamma\left[\frac{1}{\sigma r} + \frac{\sigma r}{\sinh^2(\sigma r)} - \frac{\sinh(\sigma r)}{4\cosh(\sigma r)}\right] \end{aligned}$$

and a direct calculation yields that the function in square brackets takes a negative value for $\sigma r = 7$. □

This concludes the proof of Theorem A.1.7 for $t \in \left(\pi, \frac{3\pi}{2}\right)$.

A.1.2 Second Case: $t \in \left[\frac{3\pi}{2}, \frac{5\pi}{3}\right)$

Then $\sigma \in \left[\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ and $-\gamma \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\right]$. Further $-\sigma\gamma < \frac{1}{2}$. Write

$$g_t(r) = \frac{g_1(r) + g_2(r) + g_3(r) + g_4(r)}{r\left(\sin^2(\gamma r) + \sinh^2(\sigma r)\right)}$$

so that

$$g_1(r) = -6\sigma\gamma\sin^2(\gamma r) \tag{A.1.12}$$

$$g_2(r) = -6\sigma\gamma\sinh^2(\sigma r) \tag{A.1.13}$$

$$g_3(r) = \sigma r\sin(\gamma r)\cos(\gamma r) \tag{A.1.14}$$

$$g_4(r) = \gamma r\sinh(\sigma r)\cosh(\sigma r) \tag{A.1.15}$$

Lemma A.1.16. *Under the above assumption on t let $r > \frac{9}{-2\gamma}$. Then $g_t(r) < 0$.*

Proof. We have

$$\begin{aligned}\frac{2}{3}g_4(r) + g_1(r) &< \left(\frac{2\gamma r}{3} - 6\sigma\gamma\right)\sinh(\sigma r)\cosh(\sigma r)\\ &< (3 - 6\sigma\gamma)\sinh(\sigma r)\cosh(\sigma r) < 0.\end{aligned}$$

Our assumptions on t imply that $-\frac{\sigma}{\gamma} > \frac{1}{\sqrt{3}}$ and so $\sigma\gamma > \frac{9}{2\sqrt{3}}$ whence by direct calculation we have $\sinh(\sigma r)\cosh(\sigma r) > 45$. So

$$\frac{1}{6}g_4(r) + g_1(r) = \frac{\gamma r}{6}\sinh(\sigma r)\cosh(\sigma r) - 6\sigma\gamma\sin^2(\gamma r) < 0.$$

Finally

$$\frac{1}{6}g_4(r) + g_3(r) = \frac{\gamma r}{6}\sinh(\sigma r)\cosh(\sigma r) + \frac{\sigma r}{2}\sin(2\gamma r) < -\frac{9}{2}\frac{\sigma r}{6} + \frac{\sigma r}{2} < 0.$$

□

Lemma A.1.17. *If $r < \frac{\pi}{-4\gamma}$ then $g_t(r) > 0$.*

Proof. Note that since $\sigma < -\gamma$ the fact that $-\gamma r < \frac{\pi}{4} < 1$ certainly implies $\sigma r < 1$.

Then

$$\begin{aligned}
g_1(r) + g_3(r) &= \underbrace{-6\sigma\gamma \sin^2(\gamma r)}_{>0} + \underbrace{\sigma r \sin(\gamma r)\cos(\gamma r)}_{<0} \\
&\geq -\sigma \sin(\gamma r)\left(6\gamma \sin(\gamma r) - r\right) \\
&\geq -\sigma \sin(\gamma r)\left(3\gamma^2 r - r\right) > 0
\end{aligned}$$

since $\gamma^2 > \frac{1}{2}$. Also

$$\begin{aligned}
g_2(r) + g_4(r) &= -6\sigma\gamma \sinh^2(\sigma r) + \gamma r \sinh(\sigma r)\cosh(\sigma r) \\
&= -\frac{\gamma}{\sigma}\sinh(\sigma r)\left(6\sigma^2 \sinh(\sigma r) - \sigma r \cosh(\sigma r)\right) \\
&\geq -\frac{\gamma}{\sigma}\sinh(\sigma r)\left(\frac{3}{2}\sinh(\sigma r) - \sigma r \cosh(\sigma r)\right) > 0
\end{aligned}$$

since $\frac{3}{2}\sinh(x) - x\cos(x) > 0$ for $x < \pi/4$. □

For the proof of the following lemma note that the denominator of (A.1.8) is always positive and (as previously) denote the numerator by $h(r)$.

Lemma A.1.18. *For $r > -\frac{\pi}{4\gamma}$ we have $g_t'(r) < 0$.*

Proof. Let $r > \frac{\pi}{4\gamma}$. Now $-\gamma > \sigma > \sqrt{3}\gamma$ so that $-\gamma r > \frac{\pi}{4}$ implies $\sigma r > \frac{\pi}{4\sqrt{3}}$. We will consider three separate case:

(a) $\gamma r > \frac{\sqrt{3}\pi}{2}$ which implies $\sigma r > \pi/2$ and $\sinh(\sigma r) > 1$. Also $\frac{1}{2} > -\sigma\gamma > \frac{\sqrt{3}}{4}$ and $\sigma > \frac{1}{2}$ so that

$$\begin{aligned}
h(r) &< -\frac{3\sqrt{3}}{2}\left(\sin^2(\gamma r) + \sinh^2(\sigma r)\right)^2 + 4(\sigma r)^2 \sinh(\sigma r)\cosh(\sigma r) \\
&< \sinh(\sigma r)\left[\underbrace{-\frac{3\sqrt{3}}{2}\sinh^2(\sigma r) + 4(\sigma r)^2\cosh(\sigma r)}_{Q(\sigma r)}\right]
\end{aligned}$$

A direct calculation verifies $Q\left(\frac{\pi}{2}\right) < -6$, moreover $Q'(x) < 0$ for $x > \frac{\pi}{2}$. Thus $h(r) < 0$ for the case (a).

(b) $\frac{\sqrt{3}\pi}{2} \geq -\gamma r \geq \frac{\pi}{2}$. Then $-2\gamma r \in (\pi, 2\pi)$, so $\sin(-2\gamma r) < 0$ and hence both summands of $h(r)$ are negative.

(c) $\frac{\pi}{2} > -\gamma r > \frac{\pi}{4}$. Then $\sigma r \in \left(\frac{\pi}{4\sqrt{3}}, \frac{\pi}{2}\right)$. Using the assumptions as in (a), plus the fact that $\sin(-2\gamma r) \leq 1$,

$$h(r) < -\frac{3\sqrt{3}}{2}\left(\sin^2(\gamma r) + \sinh^2(\sigma r)\right)^2 + 2(\sigma r)^2 \sinh(2\sigma r).$$

From the fact that $\sin(\cdot)$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and that $-\gamma r > \sigma r$ it follows that $\sin^2(\gamma r) > \sin^2(\sigma r)$ and so

$$h(r) < -\frac{3\sqrt{3}}{2}\left(\sin^2(\sigma r) + \sinh^2(\sigma r)\right)^2 + 2(\sigma r)^2 \sinh(2\sigma r)$$

and it is straightforward to verify that this is negative for $\sigma r \in \left(\frac{\pi}{4\sqrt{3}}, \frac{\pi}{2}\right)$.

□

Thus for the Second Case we have:

- $g_t(r) > 0$ for $r < \frac{\pi}{-4\gamma}$
- $g_t(r) < 0$ for $r > \frac{9}{2\gamma}$
- $g_t'(r) < 0$ for $r > \frac{\pi}{-4\gamma}$.

which establishes Theorem A.1.7 for this Case.

A.1.3 Third, and Last Case: $t \in \left[\frac{5\pi}{3}, 2\pi\right)$

Here we have $\sigma \in \left(0, \frac{1}{2}\right)$ and $-\gamma \in \left(\frac{\sqrt{3}}{2}, 1\right)$. Throughout this section we retain notation $g_1(r), \ldots, g_4(r)$ from (A.1.12-A.1.15) and h from (A.1.10).

Lemma A.1.19. *Under these conditions we have $g_t'(r) < 0$ for $r < \frac{3}{-\gamma}$.*

Proof. We split this into subcases:

(a) $-\gamma r \in \left[\frac{\pi}{2}, 3\right]$: then $-2\gamma r \in (\pi, 2\pi)$ so that $\sin(-2\gamma r) < 0$ and hence both summands of h are negative.

(b) $-\gamma r \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$: This is the most complicated subcase and requires further splitting.

b i) If $\sigma \in (0, 1/3)$ then $-\gamma \in \left(\frac{\sqrt{8}}{3}, 1\right)$ and so, using the fact that if $\alpha \in (0,1)$ and $x > 0$ then $\sinh(\alpha x) < \alpha \sinh(x)$ and $\sinh(2x) = 2\sinh(x)\cosh(x)$,

$$\begin{aligned} h(r) &< 6\sigma\gamma\sin^4(\gamma r) + \frac{r^2}{2}\sin(-2\gamma r)\sinh(2\sigma r) \\ &< 6\sigma\gamma\sin^4(\gamma r) + \frac{r^2}{2}\sin(-2\gamma r)\frac{\sigma}{-\gamma}\sinh(-2\gamma r) \\ &= -\frac{\sigma}{\gamma^3}\left(-6\gamma^4\sin^4(\gamma r) + \frac{(\gamma r)^2}{2}\sin(-2\gamma r)\sinh(-2\gamma r)\right) \\ &< -\frac{\sigma}{\gamma^3}\left(-\frac{128}{27}\sin^4(\gamma r) + \frac{(\gamma r)^2}{2}\sin(-2\gamma r)\sinh(-2\gamma r)\right) \end{aligned}$$

and we can verify directly that this is negative for $-\gamma r \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

b ii) If $\sigma \in [1/3, 1/2)$ then $\sigma > \frac{-\gamma}{3}$ and so

$$\begin{aligned} h(r) &< 6\sigma\gamma\left(\sin^2(\gamma r) + \sinh^2\left(\frac{\gamma r}{3}\right)\right)^2 + \frac{r^2}{2}\sin(-2\gamma r)\sinh(2\sigma r) \\ &\le -\frac{\sigma}{\gamma^3}\left[-6\gamma^4\left(\sin^2(\gamma r)) + \sinh^2\left(\frac{\gamma r}{3}\right)\right)^2 + \frac{(\gamma r)^2}{2}\sin(-2\gamma r)\sinh(2\gamma r)\right] \\ &< -\frac{\sigma}{\gamma}\left[-\frac{27}{8}\left(\sin^2(\gamma r)) + \sinh^2\left(\frac{\gamma r}{3}\right)\right)^2 + \frac{(\gamma r)^2}{2}\sin(-2\gamma r)\sinh(2\gamma r)\right] \end{aligned}$$

which can be easily shown to take negative values only for $-\gamma r \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

(c) $-\gamma r \in \left(0, \frac{\pi}{4}\right]$. Then

$$h(r) < -\frac{\sigma}{\gamma^3}\left[-\frac{27}{8}\sin^4(\gamma r) + \frac{(\gamma r)^2}{2}\sin(-2\gamma r)\sinh(2\gamma r)\right]$$

which takes negative values only for $-\gamma r \in \left(0, \frac{\pi}{4}\right)$.

This completes the proof of Lemma A.1.19 $\square$

Lemma A.1.20. *Under the conditions of this Case and for* $r \geq \frac{3}{-\gamma}$ *we have* $g_t(r) < 0$.

Proof. The proof comes in various subcases.

(a) $\sigma \in \left(0, \frac{1}{15}\right]$. Then certainly $-\sigma\gamma < \frac{1}{15}$ and

$$\begin{aligned}\frac{2}{15}g_4(r) + g_2(r) &= \frac{2\gamma r}{15}\sinh(\sigma r)\cosh(\sigma r) - 6\sigma\gamma\sinh^2(\sigma r)\\ &\leq \sinh^2(\sigma r)\left(\frac{2}{5} - 6\sigma\gamma\right) < 0\end{aligned}$$

so that

$$\begin{aligned}g(r) &\leq \frac{13}{15}g_4(r) + g_1(r) + g_2(r)\\ &< -\frac{13}{5}\sinh(\sigma r)\cosh(\sigma r) - 6\sigma\gamma\sin^2(-\gamma r) + \frac{\sigma r}{2}\sin(-2\gamma r)\\ &= -2\sigma\sinh(\sigma r)\cosh(\sigma r) - 6\sigma\gamma\sin^2(-\gamma r)\\ &\quad - \frac{3}{5}\sinh(\sigma r)\cosh(\sigma r) + \frac{\sigma r}{2}\sin(-2\gamma r)\\ &< -2\sigma r + 6\sigma - \frac{3}{5}\sigma r + \frac{\sigma r}{2} < \frac{-6}{\gamma}\sigma + 6\sigma - \frac{3}{5}\sigma r + \frac{\sigma r}{2} < 0\end{aligned}$$

(b) $\sigma \in \left(\frac{1}{15}, \frac{1}{2}\right)$ and $-\gamma r \in [3, \pi]$. Then $\sigma r > \frac{3}{\sqrt{224}} > \frac{1}{5}$ and $-\sigma\gamma < \frac{\sqrt{3}}{4}$. Therefore

$$g(r) < \left(\frac{3\sqrt{3}}{2} - 3\right)\sinh(\sigma r)\cosh(\sigma r) + \frac{3\sqrt{3}}{2}\sin^2(3) + \frac{\sigma r}{2}\sin(6) < 0.$$

(c) $\sigma \in \left[\frac{1}{5}, \frac{1}{2}\right)$ and $-\gamma r \in \left(\pi, \frac{3\pi}{2}\right]$. Then $-\frac{\sigma}{\gamma} \in \left[\frac{1}{\sqrt{24}}, \frac{1}{\sqrt{3}}\right)$ and hence $\sigma r \in \left[\frac{\pi}{\sqrt{24}}, \frac{\sqrt{3}\pi}{2}\right)$. It follows that $\sinh(\sigma r)\cosh(\sigma r) > \frac{5}{4}\sigma r$ and so

$$\begin{aligned} g(r) &< \sigma\left[-6\gamma\sin^2(\gamma r) + \left(\frac{3\sqrt{3}}{2} + \gamma r\right)\cdot\frac{5}{4}r + r\sin(\gamma r)\cos(\gamma r)\right] \\ &= -\frac{\sigma}{\gamma}\left[6\gamma^2\sin^2(\gamma r) - \frac{5}{4}\left(\frac{3\sqrt{3}}{2} + \gamma r\right)\gamma r - \gamma r\sin(\gamma r)\cos(\gamma r)\right] \\ &< -\frac{\sigma}{\gamma}\left[\frac{144}{25}\sin^2(\gamma r) - \frac{5}{4}\left(\frac{3\sqrt{3}}{2} + \gamma r\right)\gamma r - \gamma r\sin(\gamma r)\cos(\gamma r)\right] < 0. \end{aligned}$$

(d) $\sigma \in \left[\frac{1}{5}, \frac{1}{2}\right)$ and $-\gamma r > \frac{3\pi}{2}$. Since $-\sigma/\gamma$ is in the same range as for (c) we have $\sigma r > \frac{3\pi}{2\sqrt{24}}$ whence

$$\begin{aligned} g(r) &< \frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}\sinh^2(\sigma r) + \frac{\sigma r}{2} - \frac{3\pi}{2}\sinh(\sigma r)\cosh(\sigma r) \\ &< \frac{3\sqrt{3}}{2} + \frac{\sigma r}{2} + 3\left(\frac{\sqrt{3}}{2} - \frac{\pi}{2}\right)\sinh(\sigma r)\cosh(\sigma r) < 0 \end{aligned}$$

(e) $\sigma \in \left[\frac{1}{15}, \frac{1}{5}\right)$ and $-\gamma r > \pi$. Then $-\sigma\gamma < \frac{\sqrt{24}}{25}$ and $\sigma r > \frac{\pi}{\sqrt{224}}$, so

$$\begin{aligned} g(r) &< -\frac{6\gamma^2\sigma r}{\gamma r} + \frac{6\sqrt{24}}{25}\sinh^2(\sigma r) + \frac{\sigma r}{2} - \pi\sinh(\sigma r)\cosh(\sigma r) \\ &< \left(\frac{6}{\pi} + \frac{1}{2}\right)\sigma r + \frac{6\sqrt{24}}{25}\sinh^2(\sigma r) - \pi\sinh(\sigma r)\cosh(\sigma r) < 0. \end{aligned}$$

□

A.2 Solving the Variational Problem

We turn to solving the variational problem to obtain the logarithm of the right edge ρ of the square norm, using the method of Lagrange multipliers. The free cumulants are given by $k_m = \frac{\zeta(2m)}{\pi^{2m}}$ so that, denoting $I(p) = -\sum_n p_n \log(p_n)$ for a probability measure p on $\mathbb{N}$ we have

$$\begin{aligned}\log \rho = & -2\log\pi \\ & + \sup\left\{\tau\left(\sum_{n=1}^{\infty} p_n \log\zeta(2n) + I(p) + I(q)\right) : m_1(p) = m_1(q) = \frac{1}{2\tau}\right\}\end{aligned}$$

Write $G(\tau, p, q) := \tau\left(\sum_{n=1}^{\infty} p_n \log\zeta(2n) + I(p) + I(q)\right)$ and define Λ by

$$\begin{aligned}\Lambda(\tau, p, q, \lambda) = G(\tau, p, q) & + \lambda_1\left(\sum_{n=1}^{\infty} p_n - 1\right) + \lambda_2\left(\sum_{n=1}^{\infty} n p_n - \frac{1}{2\tau}\right) + \lambda_3\left(\sum_{n=1}^{\infty} q_n - 1\right) \\ & + \lambda_4\left(\sum_{n=1}^{\infty} n q_n - \frac{1}{2\tau}\right).\end{aligned}$$

Since the rate function of the underlying LDP is convex it is enough to find a unique critical point for Λ. The equations $\frac{\partial\Lambda}{p_n} = \frac{\partial\Lambda}{q_n} = 0$ yield

$$p_n = \frac{1}{Z_p}\zeta(2n)e^{n\lambda_2/\tau} \tag{A.2.1}$$

$$q_n = \frac{1}{Z_q}e^{n\lambda_4/\tau}. \tag{A.2.2}$$

Moreover equating the derivative of Λ with respect to τ with zero we obtain

$$0 = \sum_{n=1}^{\infty} p_n \log\zeta(2n) + I(p) + I(q) + \frac{\lambda_2 + \lambda_4}{2\tau^2}. \tag{A.2.3}$$

On the other hand we can evaluate the entropies $I(p)$, $I(q)$ of p and q, using equations (A.2.1) and (A.2.2):

$$I(p) = -\log Z_p + \sum_{n=1}^{\infty} p_n \log\zeta(2n) + \frac{\lambda_2}{2\tau^2} \tag{A.2.4}$$

$$I(q) = -\log Z_q + \frac{\lambda_4}{2\tau^2}. \tag{A.2.5}$$

Adding equations (A.2.4) and (A.2.5) and substituting into (A.2.3) we obtain $\log Z_p + \log Z_q = 0$ or $Z_p Z_q = 1$. Further we can express the function to be maximised in terms of λ_2, λ_4 and τ as follows:

$$G(\tau, p, q) = -\frac{\lambda_2 + \lambda_4}{2\tau}. \tag{A.2.6}$$

Solving the equations $\frac{\partial \Lambda}{\partial \lambda_j}$ for $j = 3, 4$ we obtain two expressions for Z_q:

$$Z_q = \sum_{n=1}^{\infty} e^{n\lambda_4/\tau} = \frac{e^{\lambda_4/\tau}}{1 - e^{\lambda_4/\tau}} \tag{A.2.7}$$

$$\frac{Z_q}{2\tau} = \sum_{n=1}^{\infty} n e^{\lambda_4/\tau} = \frac{e^{\lambda_4/\tau}}{(1 - e^{\lambda_4/\tau})^2}. \tag{A.2.8}$$

Combining (A.2.7) and (A.2.8) yields $2\tau = 1 - e^{\lambda_4/\tau}$ or $\lambda_4/\tau = \log(1 - 2\tau)$. Hence,

$$Z_p = \frac{1}{Z_q} = e^{-\lambda_4/\tau} - 1 = \frac{1}{1 - 2\tau} - 1 = \frac{2\tau}{1 - 2\tau}. \tag{A.2.9}$$

But we can also express Z_p as the partition function for p:

$$\begin{aligned} Z_p &= \sum_{n=1}^{\infty} \zeta(2n)\, e^{\lambda_2 n/\tau} = \sum_{n,k} n^{-2k}\, e^{\lambda_2 n/\tau} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{e^{\lambda_2/\tau}}{n^2} \right)^k \\ &= \sum_{n=1}^{\infty} \frac{e^{\lambda_2/\tau}}{n^2 - e^{\lambda_2/\tau}} = \frac{1 - \cot\gamma}{2} \end{aligned} \tag{A.2.10}$$

where $\gamma = \pi e^{\lambda_2/\tau}$. Further,

$$\frac{Z_p}{2\tau} = \sum_{n=1}^{\infty} n\zeta(2n) e^{\lambda_2/\tau} = \frac{\gamma^2 + \gamma^2 \cot^2\gamma - \gamma\cot\gamma}{4}. \tag{A.2.11}$$

Combining equations (A.2.11) and (A.2.9) we obtain

$$\frac{1}{1-2\tau} = \frac{\gamma^2 + \gamma^2 \cot^2 \gamma - \gamma \cot \gamma}{4}. \tag{A.2.12}$$

Moreover using (A.2.10) and (A.2.9) we get

$$\gamma \cot \gamma = 1 - \frac{4\tau}{1-2\tau} = \frac{1-6\tau}{1-2\tau}. \tag{A.2.13}$$

Putting this into (A.2.12):

$$\frac{1}{1-2\tau} = \frac{\gamma^2 + \left(\frac{1-6\tau}{1-2\tau}\right)^2 - \frac{1-6\tau}{1-2\tau}}{4} \tag{A.2.14}$$

and rearranging:

$$\gamma^2 = \frac{4\left(1-\tau-6\tau^2\right)}{\left(1-2\tau\right)^2}. \tag{A.2.15}$$

Since $1-\tau-6\tau^2 = 6\left(\frac{1}{3}-\tau\right)\left(\tau+\frac{1}{2}\right)$ the right-hand side of (A.2.14) is positive for $\tau \in \left[-\frac{1}{2}, \frac{1}{3}\right]$. Since τ is the inverse of twice the mean of a probability measure it must be positive. Hence the maximiser (p^*, q^*, τ^*) has $\tau^* \in (0, \frac{1}{3}]$. Putting (A.2.15) back into (A.2.13):

$$2\sqrt{1-\tau-6\tau^2} \cot\left(\frac{2\sqrt{1-\tau-6\tau^2}}{1-2\tau}\right) = 1-6\tau \tag{A.2.16}$$

which has a unique solution τ^* on $(0, \frac{1}{3})$. The above equations determine the maximiser elements p^*, q^*, and then

$$\log \rho = 2G(\tau^*, p^*, q^*) - 2\log \pi = \log\left(1-2\tau^*\right) - \log(4) - \log\left(1-\tau-6\left(\tau^*\right)^2\right). \tag{A.2.17}$$

Re-writing this in terms of $m^* = \frac{1}{2\tau^*}$ yields the equations given in the paper.

Bibliography

[1] ABRAMOWITZ, M., AND STEGUN, I. A., Eds. *Handbook of Mathematical Functions*. Dover Publications, 1965.

[2] ANDERSON, G. W., GUIONNET, A., AND ZEITOUNI, O. *An Introduction to Random Matrix Theory*, vol. 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2010.

[3] ARMSTRONG, D. *Generalized Non-Crossing Partitions and Combinatorics of Coxeter Groups*. PhD thesis, Cornell University, 2006.

[4] ASMUSSEN, S. *Applied Probability and Queues*. Wiley, New York, 1987.

[5] AUBRUN, G. Partial Transposition of Random States and Non-Centered Semicircular Distributions. *arXiv:1011.0275* (2011).

[6] AVAL, J.-C. Multivariate fuss-catalan numbers. *Discrete Math. 308* (2008), 4660 – 4669.

[7] BANICA, T., AND NECHITA, I. Asymptotic Eigenvalue Distributions of Block-transposed Wishart Matrices. *arXiv:1105.2556* (2011).

[8] BARNDORFF-NIELSEN, O. E., AND THORBJØRNSEN, S. Lévy Laws in Free Probability. *Proc. Natl. Acad. Sci. USA 99*, 26 (2002), 16576 – 16580.

[9] BARNDORFF-NIELSEN, O. E., AND THORBJØRNSEN, S. Self-decomposability and Lévy Processes in Free Probability. *Bernoulli 8*, 3 (2002), 326 – 366.

[10] BERCOVICI, H. Series of free random variables. *Journal of Theoretical Probability 18* (2005), 957 – 965.

[11] BERCOVICI, H., AND PATA, V. The Law of Large Numbers for Free Indentically Distributed Random Variables. *Ann. Prob. 24* (1996), 453–465.

[12] BERCOVICI, H., AND PATA, V. Stable Laws and Domains of Attraction in Free Probability Theory. With an appendix by Philippe Biane. *Ann. Math. 149* (1999), 1023 – 1060.

[13] BERCOVICI, H., AND VOICULESCU, D. Free convolution of measures with unbounded support. *Indiana University Mathematics Journal 42* (1993), 733 – 773.

[14] Bercovici, H., and Voiculescu, D. V. Lèvy-hinčin Type Theorems for Multiplicative and Addititve Free Convolution. *Pacific Journal of Mathematics 153* (1992), 217 – 248.

[15] Biane, P. Free Brownian Motion, Free Stochastic Calculus and Random Matrices. In *Free Probability* (1995), D.-V. Voiculescu, Ed., vol. 12 of *Fields Institute Communications*, pp. 1–19.

[16] Biane, P. On the Free Convolution with a Semi-circular Distribution. *Indiana University Mathematics Journal 46*, 3 (1997), 705 – 718.

[17] Biane, P. Some Properties of Crossings and Partitions. *Discrete Mathematics 175* (1997), 41 – 53.

[18] Biane, P. Parking Functions of Types A and B. *Electronic Journal of Combinatorics 9* (2002), N7: 1 – 5.

[19] Biane, P. Free probability for probabilists. *Quantum Probability Communications XI* (2003).

[20] Biane, P. Matrix-Valued Brownian Motion and a Paper by Pòlya. *Sèminaires de Probabilitès XLII* (2009), 171 –185.

[21] Biane, P., Bougerol, P., and O'Connell, N. Littelmann paths and Brownian paths. *Duke Math. J. 230*, 1 (2005), 127 – 167.

[22] Biane, P., Bougerol, P., and O'Connell, N. Continuous crystals and Duistermaat-Heckman measure for Coxeter groups. *Adv. Math. 221* (2009), 1522 – 1583.

[23] Biane, P., and Speicher, R. Stochastic Calculus with Respect to Free Brownian Motion and Analysis on Wigner Space. *Prob. Theory Rel. Fields 112* (1998), 373–409.

[24] Blower, G. *Random Matrices: High Dimensional Phenomena*, vol. 367 of *LMS Lecture Note Series*. Cambridge University Press, 2009.

[25] Bollobás, B. *Linear Analysis*, second ed. Cambridge University Press, 1999.

[26] Borodin, A., and Corwin, I. Macdonald processes. *arXiv:1111.4408* (2011).

[27] Boucheron, S., Gamboa, F., and Léonard, C. Bins and Balls: Large Deviations of the Empirical Occupancy Process. *Annals of Applied Probability 12*, 2 (2002), 607 – 636.

[28] Brémaud, P. *Point Processes and Queues: Martingale Dynamics*. Springer, Berlin, 1981.

[29] Burke, P. J. The Output of a Queueing System. *Operations Research 4*, 6 (1956), 699 – 704.

[30] Callan, D. Sets, lists and noncrossing partitions. *Journal of Integer Sequences 11*, Article 08.1.3 (2008), 1 – 7.

[31] Capitaine, M., and Donati-Martin, C. The Lévy Area Process for the Free Brownian Motion. *Journal of Functional Analysis 179* (2001), 153–169.

[32] Capitaine, M., and Donati-Martin, C. Free Wishart Processes. *J. Theoret. Probab. 18*, 2 (2005), 413–438.

[33] Corwin, I., O'Connell, N., Seppäläinen, T., and Zygouras, N. Tropical combinatorics and Whittaker functions. *arXiv:1110.3489* (2011).

[34] de Zelicourt, C. Une méthode de martingales pour la convergence d'une suite de processus de sauts markoviens vers une diffusion associée à une condition frontière. application aux systèmes de files d'attente. *Ann. Inst. Henri Poincaré Probab. Stat. 17*, 4 (1981), 351 – 376.

[35] Dembo, A., Vershik, A., and Zeitouni, O. Large Deviations for Integer Partitions. *Markov Processes and Related Fields 6* (2000), 147 – 179.

[36] Dembo, A., and Zajic, T. Large Deviations: From Empirical Mean and Measure to Partial Sums Process. *Stochastic Processes and Their Applications 57* (1995), 191 – 224.

[37] Dembo, A., and Zeitouni, O. *Large Deviations. Techniques and Applications*, vol. 38 of *Applications of Mathematics*. Springer, New York, 1998.

[38] den Hollander, F. *Large Deviations*, vol. 14 of *Fields Institute Monographs*. AMS, 2000.

[39] Deuschel, J.-D., and Stroock, D. W. *Large Deviations*. Academic Press, Boston, MA., 1989.

[40] Deutsch, E. Dyck path enumeration. *Discrete Mathematics 204* (1999), 167 – 202.

[41] Dufresne, D. An affine property of the reciprocal Asian option process. *Osaka J. Math 38* (2001), 379–381.

[42] Dufresne, D. The integral of geometric Brownian motion. *Adv. Appl. Prob. 33* (2001), 223–241.

[43] Durrett, R. *Probability: Theory and Examples*. Wadsworth, 1991.

[44] Edelman, P. H. Chain Enumeration and Non-Crossing Partitions. *Discrete Mathematics 31* (1980), 171 – 180.

[45] Ellis, R. S. *Entropy, Large Deviations, and Statistical Mechanics*, vol. 271 of *Grundlehren der mathematischen Wissenschaften*. Springer, New York, 1985.

[46] Francesco, P. D., Golinelli, O., and Guitter, E. Meander, Folding and Arch Statistics. *Math.Comput.Modelling 26N8 26* (1997), 97–147.

[47] Gard, T. C. *Introduction to Stochastic Differential Equations.* Marcel Dekker, 1988.

[48] Greven, A., Keller, G., and Warnecke, G. *Entropy.* Princeton University Press, 2003.

[49] Guionnet, A. *Large Random Matrices: Lectures on Macroscopic Asymptotics.* École d'été de Probabilités de St-Flour. Lecture Notes in Mathematics. Springer, 2009.

[50] Harrison, J. M. The diffusion approximation for tandem queues in heavy traffic. *Adv. Appl. Prob. 10* (1978), 886 – 905.

[51] Harrison, J. M. *Brownian Motion and Stochastic Flow Systems.* Wiley, New York, 1985.

[52] Harrison, J. M., and Reiman, M. I. Reflected Brownian motion in an orthant. *Ann. Probab. 9*, 2 (1981), 302 – 308.

[53] Harrison, J. M., and Williams, R. J. Multidimensional reflected Brownian motions having exponential stationary distributions. *Ann. Probab. 15*, 1 (1987), 115 – 137.

[54] Harrison, J. M., and Williams, R. J. On the quasireversibility of a multiclass Brownian service station. *Ann. Probab. 18* (1990), 1249 – 168.

[55] Hiai, F., and Petz, D. *The Semicircle Law, Free Random Variables and Entropy*, vol. 77 of *Mathematical Surveys and Monographs.* American Mathematical Society, 2000.

[56] Kac, M. On Some Connections Between Probability Theory and Differential and Integral Equations. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (1950), 189 – 215.

[57] Karatzas, I., and Shreve, S. E. *Brownian Motion and Stochastic Calculus*, second ed. No. 113 in Graduate Texts in Mathematics. Springer, 1998.

[58] Kardar, M., Parisi, G., and Zhang, Y.-C. Dynamic scaling of growing interfaces. *Phys. Rev. Lett. 56*, 9 (1986), 889 – 892.

[59] Kelly, F. P. *Reversibility and Stochastic Networks.* Wiley, New York, 1979.

[60] Kreweras, G. Sur les partitions non croisées d'un cycle. *Discrete Mathematics 1*, 4 (1972), 333 –350.

[61] Kümmerer, B., and Speicher, R. Stochastic Integration on the Cuntz Algebra o_∞. *Journal of Functional Analysis 103* (1992), 372 – 408.

[62] Lando, S. K., and Zvonkin, A. Plane and Projective Meanders. *Theoretical Computer Science 117* (1993), 227 – 241.

[63] Lévy, P. Wiener's Random Function, and Other Laplacian Random Functions. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (1950), 171–187.

[64] Li, L., Tulino, A. M., and Verdú, S. Asymptotic EIgenvalue Moments for Linear Multiuser Detection. *Communications in Information and Systems 1*, 3 (2001), 273 – 304.

[65] Lynch, J., and Sethuraman, J. Large Deviations for Processes with Independent Increments. *Annals of Probability 15*, 2 (1987), 610 – 627.

[66] Lyons, T. Differential Equations Driven by Rough Signals. *Revista Matématica Iberoamericana 14* (1998), 215–310.

[67] Maassen, H. Addition of Freely Independent Random Variables. *Journal of Functional Analysis 106* (1992), 409 – 438.

[68] Matsumoto, H., and Yor, M. A version of Pitmans $2M - X$ theorem for geometric Brownian motions. *C. R. Acad. Sci. Paris 328* (1999), 1067 – 1074.

[69] Matsumoto, H., and Yor, M. An analogue of Pitman's $2M - X$ theorem for exponential Wiener functionals. Part I: A time-inversion approach. *Nagoya Mathematics Journal 159* (2000), 125 – 166.

[70] Matsumoto, H., and Yor, M. A Relationship Between Brownian Motions with Opposite Drifts via Certain Enlargements of the Brownian Filtration. *Osaka J. Math. 38* (2001), 383–398.

[71] Matsumoto, H., and Yor, M. An analogue of Pitman's $2M - X$ theorem for exponential Wiener functionals. Part II: The role of the Generalized Inverse Gaussian laws. *Nagoya Mathematics Journal 162* (2001), 65 – 86.

[72] McCammond, J. Noncrossing partitions in suprising locations. *American Mathematical Monthly 113* (2006), 598 – 610.

[73] McKean, H. P. *Stochastic Integrals*. Probability and Mathematical Statistics. Academic Press, New York, 1969.

[74] Mehta, M. L. *Random Matrices*, third ed., vol. 142 of *Pure and Applied Mathematics Series*. Elsevier, 2004.

[75] Mogulskii, A. A. Large deviations for trajectories of multi dimensional random walks. *Theory Probab. Appl. 21* (1976), 300 – 315.

[76] Moriarty, J., and O'Connell, N. On the free energy of a directed polymer in a Brownian environment. *Markov Proc. Rel. Fields 13* (2007), 251 – 266.

[77] Müller, R. R., Guo, D., and Moustakas, A. L. Vector precoding in wireless communications: a replica symmetric analysis. In *Proceedings of the 2nd international conference on Performance evaluation methodologies and tools* (ICST, Brussels, Belgium, Belgium, 2007), ValueTools '07, ICST (Institute for Computer Sciences, Social-Informatics and Telecommunications Engineering), pp. 38:1–38:10.

[78] Murphy, G. J. *C*-Algebras and Operator Theory.* Academic Press, 1990.

[79] Nadakuditi, R. R., and Benaych-Georges, F. The Breakdown Point of Signal Subspace Estimation. In *Sensor Array and Multichannel Signal Processing Workshop (SAM)* (2010), IEEE, pp. 177 – 180.

[80] Neu, P., and Speicher, R. Rigorous mean-field theory for coherent potential approximation: Anderson model with free random variables. *Journal of Statistical Physics 80* (1995).

[81] Nica, A., and Speicher, R. On the Multiplication of Free n-Tuples of Noncommutative Random Variables. *Amer. J. Math. 118* (1996), 799 – 837.

[82] Nica, A., and Speicher, R. Commutators of Free Random Variables. *Duke Mathematical Journal 92*, 3 (1998), 553 – 592.

[83] Nica, A., and Speicher, R. *Lectures on the Combinatorics of Free Probability.* Cambridge University Press, 2006.

[84] O'Connell, N. Directed polymers and the quantum toda lattice. *Ann. Probab.*, To appear. arXiv:0910.0069.

[85] O'Connell, N. Directed Percolation and Tandem Queues. DIAS Technical Report DIAS-APG-9912, 1999.

[86] O'Connell, N. Whittaker functions and related stochastic processes. *arXiv:1201.4849* (2012).

[87] O'Connell, N., and Ortmann, J. Product-form invariant measures for Brownian motion with drift satisfying a skew-symmetry type condition. *arXiv:1201.5586* (2012).

[88] O'Connell, N., and Warren, J. A multi-layer extension of the stochastic heat equation. arXiv:1104.3509.

[89] O'Connell, N., and Yor, M. Brownian analogues of Burke's theorem. *Stoch. Proc. Appl. 96* (2001), 285 – 304.

[90] Ortmann, J. Functionals of the Free Brownian Bridge. arXiv:1107.0218.

[91] Ortmann, J. Large deviations for non-crossing partitions. arXiv:1107.0208.

[92] Pal, S., and Pitman, J. One-dimensional Brownian particle systems with rank-dependent drifts. *Ann. Appl. Probab. 18*, 6 (2008), 2179 – 2207.

[93] Penner, R., and Waterman, M. S. Spaces of RNA secondary structures. *Advances of Mathematics 101* (1993), 31 – 43.

[94] Prodinger, H. A correspondence between ordered trees and noncrossing partitions. *Discrete Mathematics 46* (1983), 205 – 206.

[95] Prolhac, S., and Spohn, H. The propagator of the atractive Delta-Bose gas in one dimension. *J. Math. Phys. 52* (2011), 122106.

[96] Reiman, M. I. Open queueing networks in heavy traffic. *Math. Op. Res. 9*, 3 (1984), 441 – 458.

[97] Reiner, V. Non-Crossing Partitions for Classical Reflection Groups. *Discrete Mathematics 177* (1997), 195 – 222.

[98] Revuz, D., and Yor, M. *Continuous Martingales and Brownian Motion*, third ed., vol. 293 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin; Heidelberg, 1999.

[99] Robert, P. *Réseaux, et files d'attente: methodes probabilistes.* Springer, Berlin, 2000.

[100] Rogers, C., and Williams, D. *Diffusions, Markov Processes and Martingales, volume 1: Foundations*, reprint of the second (1994) ed. Cambridge Mathematical Library. Cambridge University Press, 2000.

[101] Rogers, C., and Williams, D. *Diffusions, Markov Processes and Martingales, volume 2:* Itô *Calculus*, reprint of the second (1994) ed. Cambridge Mathematical Library. Cambridge University Press, 2000.

[102] Schied, A. Cramér's Condition and Sanov's Theorem. *Statistics and Probability Letters 39*, 1 (1998), 55 – 60.

[103] Seppäläinen, T. Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab. To appear* (2009), arXiv:0911.2446.

[104] Seppäläinen, T., and Valkó, B. Bounds for scaling exponents for a 1+1 dimensional directed polymer in a Brownian environment. *Alea 7* (2010), 451 – 476.

[105] Simion, R. Noncrossing partitions. *Discrete Mathematics 217* (2000), 367 – 409.

[106] Speicher, R. A New Example of 'Independence' and 'White Noise'. *Probability Theory and Related Fields 84* (1990), 141 – 159.

[107] Speicher, R. Multiplicative functions on the lattice of non-crossing partitions and free convolution. *Math. Ann. 298* (1994), 611–628.

[108] Speicher, R. Free Probability Theory and Non-Crossing Partitions. *Séminaire Loth. Comb. B93c* (1997).

[109] Speicher, R. *Combinatorial Theory of the Free Product with Amalgamation and Operator-valued Free Probability Theory*, vol. 627 of *Memoirs of the AMS.* 1998.

[110] Speicher, R., and Neu, P. Physical applications of freeness. In *XIIth International Congress of Mathematical Physics* (1999), D. W. et. al., Ed., International Press, pp. 261 – 266.

[111] Spohn, H. KPZ scaling theory and the semi-discrete polymer model. *arXiv:1201.0645* (2012).

[112] Stanley, R. P. *Enumerative Combinatorics, Volume 1*, vol. 49 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, 1997.

[113] Stanley, R. P. Parking Functions and Non-Crossing Partitions. *Electron. J. Combin. 4* (1997), R20.

[114] Stanley, R. P. *Enumerative Combinatorics, Volume 2*, vol. 62 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, 1999.

[115] Stroock, D. W., and Varadhan, S. R. Diffusion processes with boundary conditions. *Comm. Pure Appl. Math. 24* (1971), 147 – 225.

[116] Thygesen, U. H. A survey of Lyapunov techniques for Stochastic Differential Equations. Tech. Rep. 18-1997, IMM, 1997.

[117] Tolmatz, L. On the Distribution of the Square Integral of the Brownian Bridge. *Annals of Probability 30*, 1 (2002), 253 – 269.

[118] Tse, D. Multiuser Receivers, Random Matrices and Free Probability. In *Proceedings of the 37th Annual Allerton Conference on Communication, Control and Computing* (Monticello, IL, 1999).

[119] Tulino, A. M., and Verdú, S. *Random Matrix Theory and Wireless Communications.* Foundations and Trends in Communications and Information Theory. now, 2004.

[120] Varadhan, S. R. S. Asymptotic probabilities and differential equations. *Comm. Pure Appl. Math. 19* (1966), 261 – 286.

[121] Vershik, A. M. Statistical Mechanics of Combinatorial Partitions and Their Limit Shapes. *Functional Analysis and Its Applications 30* (1996), 19 – 30.

[122] Victoir, N. Lévy Area for the Free Brownian Motion: Existence and Non-Existence. *J. Funct. Anal. 208* (2004), 107 – 121.

[123] Voiculescu, D. Symmetries of Some Reduced Free Product C*-Algebras. *Lecture Notes in Mathematics 1132* (1985), 556 – 588.

[124] Voiculescu, D. Addition of Certain Noncommuting Random Variables. *Journal of Functional Analysis 66* (1986), 323 – 346.

[125] Voiculescu, D. Multiplication of Certain Noncommuting Random Variables. *Journal of Operator Theory 18* (1987), 223 – 235.

[126] Voiculescu, D. V. Limit Laws for Random Matrices and Free Products. *Invent. Math. 104* (1991), 201–220.

[127] Voiculescu, D. V. *Lectures on Free Probability Theory*. No. 1738 in Lecture Notes in Mathematics (Lectures on Probability and Theory and Statistics). Springer, 2000, pp. 283 – 349.

[128] Voiculescu, D. V., Dykema, K. J., and Nica, A. *Free Random Variables*, vol. 1 of *CRM Monograph Series*. American Mathematical Society, 1992.

[129] Wigner, E. P. On the Distribution of the Roots of Certain Symmetric Matrices. *Annals of Mathematics 67* (1958), 325 – 327.

[130] Williams, R. J. Reflected Brownian motion with skew symmetric data in a polyhedral domains. *Probab. Theory Relat. Fields 75* (1987), 459 – 485.

[131] Yano, F., and Yoshida, H. Some set partition statistics in non-crossing partitions and generating functions. *Discrete Mathematics 307* (2007), 3147 – 3160.